BELFER GRADUATE SCHOOL OF SCIENCE
MONOGRAPHS SERIES
NUMBER THREE

Lectures on Quantum Field Theory

PAUL A. M. DIRAC

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by
P. A. M. DIRAC

Published by
Belfer Graduate School of Science
Yeshiva University
New York
1966

QC 174.45.D55

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Produced and distributed by
ACADEMIC PRESS INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition distributed by
ACADEMIC PRESS INC. (LONDON) LTD.
Berkeley Square House, London W.1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 66-20943

First Printing, 1966
Second Printing, 1967

PRINTED IN THE UNITED STATES OF AMERICA

Preface

The treatment of quantum field theory given here is based on the Hamiltonian formalism and the theory is developed as far as possible in a logical manner, with results following from equations of motion. This involves a considerable departure from the usual treatment of quantum field theory in which one rather abandons the hope of logical deduction and is content with setting up working rules and showing their consistency, the equations of motion getting lost in the process.

In physics one should aim at a comprehensive scheme for the description of the whole of nature. A vast domain in physics can be successfully described in terms of equations of motion. It is necessary that quantum field theory be based on concepts and methods that can be unified with those used in the rest of physics. This necessity forces one to think of quantum field theory in terms of equations of motion. The usual treatment should thus be considered a stopgap, without any lasting future.

These lectures were given at Yeshiva University during the academic year 1963–1964. The tape recording of the lectures was transcribed by Daniel Wisnivesky, to whom I am greatly indebted for giving so much time and care to the work. I have made a number of alterations to present the material in a form suitable for a book.

P. A. M. DIRAC

Cambridge, England
August 1966

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§1. *Relationship of the Heisenberg and Schrödinger Pictures*

In atomic theory we have fields and we have particles. The fields and the particles are not two different things. They are two ways of describing the same thing—two different points of view. We use one or the other according to convenience. In general the particle point of view is the more convenient one for problems involving only a few particles. It is the one usually used for elementary problems. When we deal with many particles of the same kind, the field point of view is the more convenient. For a general theory, which is to be formulated accurately, we have many particles coming in, maybe an infinite number, and the field concepts are useful. The present course of lectures will be concerned with general theory, so it will be based mainly on the field point of view. I shall assume that you have some knowledge of the basic ideas of quantum theory and also that you are familiar with relativity and Maxwell's electrodynamics.

Our object is to get a single comprehensive theory that will describe the whole of physics. This theory should consist of a scheme of equations, together with the rules for applying and interpreting the equations. The equations by themselves do not form a physical theory. Only when they are joined onto the rules for using them do we have a physical theory. A primary requirement is that the equations and the rules for using them should be *consistent.*

We need a single consistent comprehensive theory. Any special theory that one sets up for dealing with a particular problem should be consistent with this general theory. I do not need to tell you that such a theory has not yet been attained. It is the ultimate goal, toward which all physicists are working. These lectures are intended to show you how far one can go toward this goal at the present time.

Quantum field theory is a subject that has been very much developed during the last fifteen years. The recent development started with the discovery of the Lamb shift and its theoretical explanation, a very brilliant discovery because of the close agreement between the theory and the experiment. Many papers have been written on the subject, and now there are quite a number of books about it, and people who want to learn about this recent work on quantum field theory will be better advised to follow some of the books instead of going through the original papers.

This work, though, has serious difficulties in it. There are many quantities appearing in the theory which turn out to be infinite although they ought to be finite; and physicists have followed all sorts of tricks to avoid these difficulties, but the result is that the theory is in rather a mess. The departures from logic are very serious and one really gives up all pretense of logical development in places. In this course of lectures I propose to follow different lines, which avoid some of the difficulties. I won't be able to avoid all of them, but I will avoid some of the worst ones and I will, at any rate, be able to preserve some semblance of logic. By that I mean that I will neglect only quantities that one can have some physical feeling to count as small, instead of doing what people often do in this kind of work: neglect something that is really infinitely great. It will not be possible for me to set things out rigorously. I shall neglect things I shan't be able to prove are really small, but still I hope you'll have some feeling that these things are small.

The system of approximations I shall use will be somewhat similar to the approximations that engineers use in their calculations. Engineers have to get results and there are so many factors occurring in their problems that they have to neglect an awful lot of them; they don't have time to study everything seriously and they develop a sort of feeling as to what can be neglected and what can't. I believe that physicists working in quantum field theory will have to develop a similar sort of feeling as to what can be neglected and what can't. The final test is whether the resulting theory is coherent and in reasonable agreement with experiment.

Well, those are the general features of the theory that I shall be presenting to you and I would like to say at the outset in what way my form of the theory differs from the usual form. In quantum mechanics as it is usually taught there are two pictures that one may use. There is Heisenberg's picture, in which one works with Heisenberg's dynamical variables, and in which one has Heisenberg's equation of motion,

$$i\hbar \frac{du}{dt} = uH - Hu. \tag{1.1}$$

H, the Hamiltonian, is the quantity that represents the total energy of the system. In the Schrödinger theory we work with state vectors that represent the states. We have a state vector representing a particular state, denoted by a symbol such as $|A\rangle$, and it varies with the time according to the equation

$$i\hbar \frac{d}{dt}|A\rangle = H|A\rangle. \tag{1.2}$$

H is the same in both pictures. The essential difference between the two pictures is that in Heisenberg's the dynamical variables vary with the time while the state vectors are fixed; in Schrödinger's the state vectors vary with the time while the dynamical variables are fixed.

The two pictures are connected in this way: any Schrödinger dynamical variable is connected with the corresponding Heisenberg dynamical variable by the transformation

$$u_{\mathrm{S}} = e^{-iHt/\hbar} u_{\mathrm{H}} e^{iHt/\hbar}, \tag{1.3}$$

and state vectors are correspondingly connected by

$$|A_{\mathrm{S}}\rangle = e^{-iHt/\hbar}|A_{\mathrm{H}}\rangle. \tag{1.4}$$

One easily checks that, with u_{S} constant, (1.3) makes u_{H} vary with t according to (1.1), and with $|A_{\mathrm{H}}\rangle$ constant, (1.4) makes $|A_{\mathrm{S}}\rangle$ vary with t according to (1.2).

People usually say that these equations establish the equivalence of the two pictures—that one can use whichever one likes indiscriminately. However, the argument is valid only if $e^{iHt/\hbar}$ exists and can be applied to one state vector to give another, and for the Hamiltonians one meets with in quantum field theory there is good reason to believe that this is not so, because of convergence difficulties, and so the two pictures are not equivalent.

The reasons for believing that the two pictures are not equivalent arise from examples for which one can find the Hamiltonian and be confident that it is correct. There is the example of quantum electrodynamics, in which one considers electrons and positrons interacting with the electromagnetic field. One can set up the Hamiltonian for that problem and be pretty confident it is correct, because when one uses it in the Heisenberg equations of motion one gets sensible field equations, which are in close analogy to the classical field equations, are relativistic and look satisfactory from every point of view. On the other hand, when one uses that Hamiltonian in the Schrödinger picture, one gets a Schrödinger equation that has no solutions (at any rate no solutions of physical value). People have been trying very hard for very many years, decades even, to get solutions and they have failed, and now it seems pretty certain that solutions do not exist. At any rate, if they do exist they are of no use to us if we cannot find them.

One would expect in any case that there should be a trivial solution corresponding to the physical state for which there are no particles and no fields present—the perfect vacuum. The vacuum is quite a trivial thing physically and we should expect it to correspond to a trivial solution of the Schrödinger equation, but there isn't even this trivial solution. That makes one think that *the Heisenberg picture is a good picture and the Schrödinger picture is a bad picture and that the two pictures are not equivalent.*

One can consider $e^{-iHt/\hbar}$ to exist in a formal way and multiply it into a constant state vector $|A\rangle$. Then the product $e^{-iHt/\hbar}|A\rangle$ will satisfy the Schrödinger equation (1.2). The Schrödinger wave function consists of the coordinates of $e^{-iHt/\hbar}|A\rangle$. The nonexistence of the Schrödinger wave function may be attributed to $e^{-iHt/\hbar}|A\rangle$ not having coordinates, rather than to its nonexistence. It lies in a space of too many dimensions to have coordinates.

The state vectors are usually supposed to be vectors in Hilbert space, in which case they would have coordinates. I should say here that I am using the term *Hilbert space* in the sense in which Hilbert originally used it—as a space with an infinite number of dimensions but still a separable space, that is, a space that can be spanned by an enumerable infinity of basic vectors. Mathematicians today use the term *Hilbert* *space* in a more general sense—to include also spaces that are not separable—but whenever I mention Hilbert space it is to be understood as a separable Hilbert space. The vectors in such a separable Hilbert space can be represented by coordinates.

We are led to the conclusion that in quantum field theory the state vectors are not in Hilbert space. We may start off a particular state vector in Hilbert space at a particular time. Suppose we then make it vary with time in accordance with the Schrödinger equation: What would happen to it? Roughly, what happens to it is that it gets knocked right out of Hilbert space in the shortest time interval possible. The interactions that are physically important in quantum field theory are so violent that they will knock any Schrödinger state vector out of Hilbert space in the shortest possible time interval. It is thus not possible to get a solution of the Schrödinger equation for which the state vector stays in Hilbert space. So the Schrödinger picture is unworkable.

The Heisenberg picture is a good picture, because the Heisenberg equations of motion are reasonable. They are not entirely free of difficulty; I am not asserting that we don't get into any difficulties at all using the Heisenberg equations in quantum electrodynamics, for instance, but we do not get into such profound difficulties as we would if we tried to base everything on an equation that doesn't have any solutions at all.

We have the problem therefore of setting up quantum field theory working entirely in the Heisenberg picture. That is what I shall try to do in this course of lectures. We may take the existing theory given in the textbooks, for instance, and then cut out all references to the Schrödinger picture, and see how well we can get on without them. That is what I would ask you to do if you are going to use the textbooks at all. Wherever you see references to the Schrödinger picture, cut them out. You will find that very often you can get on fairly well without these references, and that by cutting them out the whole theory becomes more logical and more intelligible. I would say that we are cutting out a lot of deadwood from the usual presentation of the quantum field theory. What is left, referring only to the Heisenberg picture, is what is essential and what we should concentrate our attention on.

We have now to survey the usual quantum theory and see how much of it survives with this present point of view. If we just consider relationships at one particular time, you see that just about everything survives. At one particular time t_0 we have dynamical variables $\alpha(t_0)$. These Heisenberg dynamical variables will operate to the right on some vectors we shall call *ket vectors.* The situation that we have at one particular time is essentially the same whether we are thinking in terms of the Heisenberg picture or the Schrödinger picture. The difference between the Heisenberg picture and the Schrödinger picture comes into play only when we vary t_0. So all the usual quantum theory, insofar as it concerns relationships between dynamical variables and kets at one particular time, can be taken over and preserved in the new formulation.

We have the kets at one particular time and we may picture them as corresponding to physical states at that time, so we retain the concept of a physical state at a certain time. We do not have the concept of a physical state throughout all time. The latter concept, which lies at the basis of the Schrödinger picture, will not occur in the present formulation. We have only the much weaker concept of a physical state at a particular time. It is a nonrelativistic concept, a very special thing which refers to one particular Lorentz frame. The Schrödinger concept of the state throughout all time would be a relativistic concept if it existed and would therefore be something one could think of as being rather fundamental in nature. But it does not exist, and all we have about states in the present formulation is a physical state at a certain time, a nonrelativistic concept—a very special thing, but useful in the calculations.

We can preserve the usual quantum relationships between dynamical variables and kets when we keep to the one time t_0. For example, take the equation $\alpha(t_0)|A\rangle = a|A\rangle$, where a is a number. If we had that equation, we could say that $|A\rangle$ represents the state at time t_0 for which the dynamical variable α at time t_0 certainly has the value a. The usual theory of eigenvalues and eigenvectors and their physical interpretation remains valid provided we stick to one time.

There is one respect, though, in which the present formulation differs from the usual one even when we think of only one time, and that is that the totality of kets which we have for our dynamical variables at one particular time to operate on do not form a Hilbert space.

We cannot get solutions in the Schrödinger picture with the kets remaining in Hilbert space, and corresponding to that we cannot get solutions in the Heisenberg picture in which our dynamical variables operate only on the kets of a Hilbert space. We have to imagine that the dynamical variables of the Heisenberg picture operate on the vectors of some space that is a bigger space than a Hilbert space. I don't know the mathematical nature of this more general space—it is better to leave it unspecified for the present than make a guess which might later turn out to be unsuitable for physical purposes. I expect the mathematicians will make guesses at what the space is, but they might very well guess wrong, and they already have made a wrong guess when they thought that Hilbert spaces were adequate for physical purposes. It is better to concentrate on the physics and try to develop a physical theory that will be successful in examples.

Question: Does the difficulty correspond entirely to the fact that there are an infinite number of degrees of freedom? Is that the sole cause?

Answer: I won't say that it is the sole cause. You might very well have an infinite number of degrees of freedom and an interaction between them which is sufficiently soft that there exists a solution of the Schrödinger equation in Hilbert space. That would be possible, but that kind of interaction is not the kind of interaction that occurs in nature. For the kind of interaction that occurs in nature, when we try to get the solution in the usual way by a perturbation method, we get integrals that do not converge. We might set up a nonrelativistic theory with a different kind of interaction for which the integrals do converge, by putting in some sort of cutoff, for instance, or just an interaction that is not so big for the high frequencies. That would be mathematically acceptable if you don't insist on relativity, and would lead to a Schrödinger equation you could solve, even though there is an infinite number of degrees of freedom with interaction between them. So the mere fact that you have an infinite number of degrees of freedom with interaction between them is not sufficient to throw out the Schrödinger picture. It is thrown out by the interactions which physicists are interested in being so violent in the high frequencies, and it doesn't seem to be possible to get interactions satisfying relativity which do not have this violence in the high frequencies.

I would agree that maybe physicists in the future will be able to think of a different Hamiltonian for which the interaction is not so violent at the high frequencies, and then the Schrödinger picture will come back into its own. That is quite a possibility. The theory I am presenting to you now I won't say is the final theory. I would only say that for the kind of Hamiltonians we work with at present, which are successful in giving results such as the Lamb shift and the anomalous magnetic moment, one cannot use the Schrödinger picture. It is better to abandon all attempts at using the Schrödinger picture with these Hamiltonians. The best quantum field theory one can formulate at the present time is one that works entirely in the Heisenberg picture; but one cannot be sure that at some future time people might not think of a new Hamiltonian for which the Schrödinger picture *will* work.

One has to preserve an open mind on these subjects if one is a research worker, and one musn't believe in anything too strongly; one must always be prepared that various beliefs one has had for a long time may be overthrown. So please understand that what I am saying is, I believe, the best that can be formulated at the present time, but I certainly don't want to assert that it will survive a long time. I don't want to assert that the Schrödinger picture will not come back; in fact, there are so many beautiful things about it that I have the feeling in the back of my mind that it ought to come back. I am really very loath to give it up. That is why it has taken so many years to think that one may have to give it up. Still one has to face the realities of the situation at the present time, and they force one to give it up for the present.

§2. *Basic Quantum Concepts: Hilbert-Space Case*

In the quantum theory our dynamical variables are quantities that satisfy noncommutative algebra. We make the fundamental assumption that the quantum variables that correspond to the classical Hamiltonian variables q_r and p_r satisfy at any time the commutation relations

$$\begin{gathered} q_r q_s - q_s q_r = 0, \qquad p_r p_s - p_s p_r = 0, \\ q_r p_s - p_s q_r = i\hbar\, \delta_{rs}. \end{gathered} \tag{2.1}$$

We define the quantum Poisson bracket of any two quantities u and v by

$$i\hbar[u, v] = uv - vu. \tag{2.2}$$

With this definition we see from (2.1) that in the cases when u and v are any two of the quantities q_r, p_s, the quantum Poisson bracket has the same value as the classical Poisson bracket.

With this definition of the quantum Poisson bracket we see that the classical Poisson-bracket laws hold also for the quantum Poisson bracket; we have in the quantum theory, as in the classical theory, the laws

$$\begin{gathered}[u, v] = -[v, u], \qquad [u_1 + u_2, v] = [u_1, v] + [u_2, v], \\ [u_1 u_2, v] = u_1[u_2, v] + [u_1, v]u_2 .\end{gathered} \tag{2.3}$$

In this equation one has to be careful to keep the order of u_1 and u_2; that is, all through the equation, u_1 should occur to the left of u_2. Finally, we have the Jacobi identity

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{2.4}$$

Now the question arises: What is the mathematical nature of the dynamical variables of quantum theory? In simple examples, one can consider them to be matrices. In fact, in Heisenberg's original quantum mechanics they were simply matrices. They are matrices usually with an infinite number of rows and columns. That is, each quantity u can be represented as a two-dimensional array of numbers, where the number of rows and columns is infinite and there is a one-to-one correspondence between the rows and columns. Such matrices can be added and multiplied according to the standard rules of matrix algebra; and matrices are noncommutative, so they can be fitted in with the scheme of equations (2.1).

Now if we look upon our dynamical variables as matrices, we may allow them to be multiplied into a column matrix, a matrix consisting of a single column, by putting this column matrix to the right of the square matrix. The column matrix can, of course, be considered as a vector: It is just a set of numbers and we may look upon the numbers as the coordinates of a vector in some vector space. The column matrices that are important physically are the ones for which the sum of the squares of the moduli of all the numbers is finite; and then the column matrix corresponds to a vector in Hilbert space. The dynamical variables are then operators in Hilbert space. Each of them can be multiplied into one of these vectors and gives another vector according to a linear law.

I should like to remind you that I shall always be using the term "Hilbert space" to denote the space in which the vectors can be represented by coordinates as above. Using the term "Hilbert space" in this sense we see that in simple examples the quantum dynamical variables can be looked upon as operators in Hilbert space.

There is another way in which we may look upon them as operators in a Hilbert space. We may consider a row matrix and we may multiply it on the left into a square matrix by matrix multiplication and get another row matrix. The elements of the row matrix may also be considered as the coordinates of a vector in a Hilbert space. Then we have our dynamical variables operating to the left on the vectors of a Hilbert space, producing other vectors in that Hilbert space. We have two Hilbert spaces, one Hilbert space with the coordinates expressed as a row matrix, the other Hilbert space with the coordinates expressed as a column matrix, and our dynamical variables can operate to the left on the first kind of Hilbert vector and to the right on the second kind of Hilbert vector.

A convenient notation is the following: The vectors on the right are called *kets*, and I write them $|A\rangle$, with a label such as A when necessary; the vectors on the left are called *bras*, and I write them as a sort of mirror image of the ket vectors $\langle B|$. The reason this notation is convenient is that we may multiply a row vector by a column vector, and then we get just a number $\langle B|A\rangle$; and this, you see, is a complete bracket expression. The bra vector is the first half of this complete bracket expression; the ket vector is the second half of this complete bracket expression; and the two of them multiplied together form a number.

We may form quantities like $\langle B|u|A\rangle$ if we have associative multiplication between our kets, bras, and dynamical variables. We shall assume associative multiplication for the quantities occurring in quantum theory. Mathematically it isn't always valid. One can construct examples of matrices for which associative multiplication doesn't hold, but these examples have not so far turned out to be important physically, and one can get on very well physically if one assumes associative multiplication.

To introduce coordinates in a general way in a Hilbert space we may introduce a set of basic kets $|r\rangle$, $r = 1, 2, \ldots, \infty$, with conjugate bras $\langle r|$, to satisfy the following conditions:

(1) They are complete; that is, any ket $|P\rangle$ can be written $|P\rangle = \sum_r P_r|r\rangle$.

(2) They are orthonormal; that is, $\langle r|s\rangle = \delta_{rs}$.

(3) They are independent; that is, there is no linear relationship among them.

(4) Closure relation $\sum_r|r\rangle\langle r| = 1$.

One can assume (1) and (2) and deduce (3) and (4), or, alternatively, assume (3) and (4) and deduce (1) and (2).

In the Hilbert space the basic kets satisfying conditions (1), (2), (3), and (4) provide a system of coordinates. Any ket $|P\rangle$ has coordinates $\langle r|P\rangle$; any bra $\langle Q|$ has coordinates $\langle Q|r\rangle$; and any dynamical variable α has coordinates $\langle r|\alpha|s\rangle$, which form a square matrix. From condition (4) we deduce the law of matrix multiplication:

$$\langle r|\alpha\beta|s\rangle = \sum_t \langle r|\alpha|t\rangle\langle t|\beta|s\rangle.$$

There are some dynamical variables which are represented by diagonal matrices:

$$\langle r|v|s\rangle = f(r)\delta_{rs}.$$

There might be several such v's, and they all commute.

In a particular physical problem we look for a complete set of commuting observables which we want to be represented by diagonal matrices. We then take their simultaneous eigenkets as basic kets:

$$|v'\rangle = |v_1', v_2', \ldots\rangle,$$

where $|v_r'\rangle$ is an eigenvector of v_r belonging to the eigenvalue v_r'.

We may introduce a second complete set $w_1, w_2, \ldots$ of commuting observables w and a corresponding set of basic kets $|w'\rangle$. A ket $|P\rangle$ will then have coordinates $\langle v'|P\rangle$ and $\langle w'|P\rangle$ in the two systems of coordinates. These are related by

$$\langle v'|P\rangle = \sum_{w'} \langle v'|w'\rangle\langle w'|P\rangle.$$

The above discussion in Hilbert space applies to simple cases and the question arises: What are the simple cases? Well, it seems that any dynamical system involving only a finite number of degrees of freedom, or any problem in which only a finite number of degrees of freedom comes into play, can be treated in this way. The quantum dynamical variables can then be looked upon as matrices or as operators on the vectors of a Hilbert space.

But for quantum field theory, that's not good enough; in quantum field theory we have an infinite number of degrees of freedom coming in and then in general it doesn't seem to be possible to look upon our dynamical variables as matrices or as operators in Hilbert space. People have done a great deal of work in quantum field theory, on the assumption that the dynamical variables are, even then, operators in Hilbert space; this work leads to a lot of difficulties, many of which can be evaded by not making the assumption.

§3. *Basic Quantum Concepts: General Case*

In the general case the question arises: What is the precise mathematical nature of the dynamical variables? It could be that they are just operators in the more general Hilbert spaces mathematicians now work with, but I'm not sure that is the case. I think it's better not to have any preconceived ideas about it or we may be led astray like people were led astray for a long time by assuming that the dynamical variables are always operators in Hilbert space. It's better, for the present, to keep an open mind about these dynamical variables and just call them *q-numbers.*

q-numbers, of course, is just a name. Ordinary numbers, when one wants to make a distinction between them and these dynamical variables, may be called *c-numbers.* The reason for the letters q and c are that q might make you think of quantum or queer, and c of classical or commuting. The essential difference between c-numbers and q-numbers is that c-numbers commute with everything, and q-numbers do not in general commute with other q-numbers.

q-numbers obey the following algebraic axioms: the ordinary axioms of addition; the distributive axiom of multiplication $u(v_1 + v_2) = uv_1 + uv_2$; and the associative axiom of multiplication $u(vw) = (uv)w$. With these axioms we can establish the quantum Poisson-bracket laws (2.3) and (2.4), in analogy to the classical Poisson-bracket laws. But we do not have commutative multiplication, and if a product uv is equal to zero, it does not follow that either of the factors equals zero. In fact, this situation occurs also with matrices. You can very well have the product of two matrices equal to zero without either of the factors being equal to zero. Well, q-numbers are just used in connection with these algebraic axioms; calculations in quantum mechanics are just calculations working out the consequences of the basic equations (2.1) with the use of noncommutative algebra.

When it's a question of forming limits, we cannot proceed with mathematical rigor. Of course, we cannot have mathematical rigor without knowing the precise mathematical nature of the quantities we are dealing with. We shall continually be using q-numbers in integrals and differentiations, and we shall be making various assumptions which seem reasonable physically about the limits we use, but we're not able to establish the theory with mathematical rigor.

We proceed to build up the theory of q-numbers by making assumptions that look reasonable as we need them. The first thing we need is some standard of reality. In the classical theory we have a clear-cut distinction between real dynamical variables and complex dynamical variables involving the root of -1, and we need some corresponding clear-cut distinction between real q-numbers and complex ones in the quantum theory. The q's and the p's we start off with in our quantum theory we assume to be real q-numbers. In the classical theory we have the general rule that the Poisson bracket of two real things is real, and we make the corresponding assumption in the quantum theory: If u and v are real, their Poisson bracket is real. It follows that $uv - vu$ equals a pure imaginary q-number except when it vanishes. We can infer that uv and vu are not both real, except possibly when u and v commute.

To preserve symmetry between the two products uv and vu, I'm going to make the assumption that $uv + vu$ is real, if u and v are real. One can check that it doesn't lead to inconsistencies; in fact, it is well known that it doesn't lead to inconsistencies for the case in which the q-numbers can be represented by matrices.

The conjugate of uv (let's write it $\overline{uv}$) is

$$\overline{uv} = \tfrac{1}{2}\overline{uv + vu} + \tfrac{1}{2}\overline{uv - vu} = \tfrac{1}{2}(uv + vu) - \tfrac{1}{2}(uv - vu) = vu.$$

We can now work with complex q-numbers. Suppose

$$\xi = u_1 + iu_2, \quad \eta = v_1 + iv_2 \qquad \text{with } u_1, u_2, v_1, v_2 \text{ real.}$$

Then the conjugate complex of the product $\xi\eta$ is

$$\begin{aligned}\overline{\xi\eta} &= \overline{(u_1 + iu_2)(v_1 + iv_2)} = v_1u_1 - v_2u_2 - iv_2u_1 - iv_1u_2 \\ &= (v_1 - iv_2)(u_1 - iu_2) = \bar{\eta}\,\bar{\xi}.\end{aligned}$$

So to take the conjugate complex of a product, you take the conjugate complex of each factor and reverse their order. One finds that the rule extends to the product of more than two factors: $\overline{\xi\eta\zeta} = \bar{\zeta}\,\bar{\eta}\,\bar{\xi}$, and so on. We now have a standard of reality for our q-numbers, and we are able to determine whether any of them is real, or what its real and imaginary parts are.

We assume that a q-number u can be multiplied into vectors on the right, called *ket vectors* and written $|A\rangle$, to give further ket vectors $u|A\rangle$, and into vectors on the left, called *bra vectors* and written $\langle B|$, to give further bra vectors $\langle B|u$. A ket $|A\rangle$ and a bra $\langle B|$ may be multiplied, with the bra on the left, to give a number $\langle B|A\rangle$.

It is no longer possible to introduce coordinates for the kets and bras. If we tried to introduce basic kets and bras in accordance with the scheme of the preceding section, we should find that we could not satisfy conditions (1) and (4), because the number of dimensions is too large. We shall develop the theory by making suitable assumptions about the kets and bras which agree as closely as possible with the situation in the Hilbert-space case.

In the Hilbert-space case each ket $|A\rangle$ with coordinates A_r has a conjugate bra for which the coordinates are the conjugate complex numbers $\bar{A}_r$, which bra is labeled by the same letter A, so it is written $\langle A|$. We have the general relationships

$$\langle B|A\rangle = \overline{\langle A|B\rangle}, \qquad \langle A|A\rangle > 0. \tag{3.1}$$

We are going to assume that in the case when the vectors are not in Hilbert space and cannot be represented by coordinates, each ket $|A\rangle$ still has a conjugate bra $\langle A|$ and the relationships (3.1) still hold.

We also assume that we have associative and distributive multiplication for q-numbers and kets and bras and that for any product of q-numbers and kets and bras we get the conjugate product by reversing the order of all the factors and taking the conjugate of each factor.

Suppose we take any q-number U with a reciprocal V. V is the reciprocal of U if and only if $UV = 1$ and $VU = 1$. Both conditions are necessary in a noncommutative algebra. If we have only $UV = 1$, we can infer $UVU = U$ or $U(VU - 1) = 0$, so either $VU = 1$ or there exists $\xi \neq 0$ such that $U\xi = 0$. It is quite possible for the product of two q-numbers to be equal to zero without either of them being zero.

Let us assume that V is the reciprocal of U; then we write $V = U^{-1}$. For any q-number α we can then set up the similarity transformation $\alpha^* = U\alpha U^{-1}$.

It is easy to see that this transformation preserves the algebraic relationships between q-numbers. The transformation does not necessarily preserve reality relations. We should like to impose further conditions on U such that the similarity transformation preserves reality relations, that is, such that when α is real so is α^*.

Suppose that $\alpha = \bar{\alpha}$ implies $\alpha^* = \bar{\alpha}^*$. Then $U\alpha U^{-1} = \alpha^* = \bar{\alpha}^* = \overline{U\alpha U^{-1}} = \overline{U^{-1}}\alpha\bar{U}$. Now $\overline{U^{-1}} = \bar{U}^{-1}$, so

$$U\alpha U^{-1} = \bar{U}^{-1}\alpha\bar{U}$$

or

$$\bar{U}U\alpha = \alpha\bar{U}U.$$

We now have that $\bar{U}U$ commutes with every q-number, because any q-number α can be split into

$$\alpha = \alpha_1 + i\alpha_2,$$

with both α_1 and α_2 real, and $\bar{U}U$ has to commute with both of those parts. A q-number that commutes with everything may be taken to be a c-number, so $\bar{U}U = k$, a c-number. It is a real c-number because $\overline{\bar{U}U} = \bar{U}U$. Furthermore, it is positive, because for any ket $|A\rangle$ we may form a new ket $U|A\rangle$ and then form $\langle A|\bar{U}U|A\rangle$, which can be looked upon as the scalar product of the ket $U|A\rangle$ with its conjugate bra, so

$$\langle A|\bar{U}U|A\rangle = k\langle A|A\rangle > 0,$$

and as $\langle A|A\rangle > 0$, we must have $k > 0$.

Take now $U' = k^{-1/2}U$; then $\bar{U}'U' = 1$. Such a U' is called a *unitary q-number*, and the transformation $\alpha^* = U'\alpha U'^{-1}$ a *unitary transformation of q-numbers.*

Let us consider now an infinitesimal unitary transformation defined by $U' = 1 + \epsilon A$, where ϵ is an infinitesimal real c-number. Then $U'^{-1} = 1 - \epsilon A$, and for any q-number α we have

$$\alpha^* = (1 + \epsilon A)\alpha(1 - \epsilon A) = \alpha + \epsilon(A\alpha - \alpha A) = \alpha - i\hbar\epsilon[\alpha, A].$$

We have the unitary condition

$$1 = \bar{U}'U' = (1 + \epsilon\bar{A})(1 + \epsilon A) = 1 + \epsilon(\bar{A} + A),$$

so $\bar{A} + A = 0$ and we can put $A = iB$, with B real. Then we can write the effect of the infinitesimal unitary transformation on α as

$$\alpha^* - \alpha = \hbar\epsilon[\alpha, B].$$

Now if we refer back to the Heisenberg equations of motion we see that the connection between a dynamical variable at time t and that variable at time $t + \delta t$ is just of this nature. For δt small we have

$$g_{t+\delta t} - g_t = \delta t[g, H].$$

In classical mechanics the connection is an infinitesimal contact transformation. The contact transformations of classical mechanics correspond to unitary transformations in the quantum theory.

There is another way of writing $g_{t+\delta t}$,

$$g_{t+\delta t} = e^{iH\delta t/\hbar} g_t e^{-iH\delta t/\hbar},$$

if we assume that

$$e^{iH\delta t/\hbar} = 1 + iH\delta t/\hbar \qquad \text{for small } \delta t.$$

We can now integrate to get the connection between the dynamical variables at two times which differ appreciably:

$$g_{t_2} = e^{iH(t_2-t_1)/\hbar} g_{t_1} e^{-iH(t_2-t_1)/\hbar}.$$

It can be proved formally that this formula is correct. However, it is not in general a useful equation, because of the difficulty of giving a meaning to the exponential. One could try to define it by a power series but one does not even know whether the series converges or not, because there is no general definition of convergence of q-numbers. In practice, to get the correction between g_{t_2} and g_{t_1}, one has to integrate the Heisenberg equations of motion.

In the present general case we cannot define a ket by coordinates. We may define a ket $|S\rangle$ by imposing suitable linear conditions on it, say

$$\xi_1|S\rangle = 0, \ \xi_2|S\rangle = 0, \ldots, \xi_r|S\rangle = 0,$$

where the ξ's are q-numbers, not necessarily real. These conditions need not be independent. They imply

$$(\xi_1\xi_2 - \xi_2\xi_1)|S\rangle = 0 \qquad \text{or} \qquad (\xi_1\xi_2 + \xi_2\xi_1)|S\rangle = 0,$$

which are the consistency conditions.

§4. *The Fock Representation*

Let us suppose that we have a harmonic oscillator with energy

$$H = \tfrac{1}{2}(p^2 + q^2).$$

Introduce the complex q-numbers

$$\eta = (2\hbar)^{-1/2}(p + iq), \qquad \bar{\eta} = (2\hbar)^{-1/2}(p - iq). \qquad (4.1)$$

Using the usual relation $qp - pq = i\hbar$, we find

$$\bar{\eta}\eta - \eta\bar{\eta} = 1.$$

By induction we have for any positive integer n,

$$\bar{\eta}\eta^n - \eta^n\bar{\eta} = n\eta^{n-1},$$

so $\bar{\eta}$ is like the operator $\partial/\partial\eta$.

We also have

$$2\hbar\eta\bar{\eta} = (p + iq)(p - iq) = p^2 + q^2 - \hbar = 2H - \hbar,$$
$$\left(\frac{H}{\hbar} - \frac{1}{2}\right)\eta = (\eta\bar{\eta})\eta = \eta(\bar{\eta}\eta) = \eta\left(\frac{H}{\hbar} + \frac{1}{2}\right),$$

or $H\eta = \eta(H + \hbar)$. We can infer

$$H\eta^n = \eta(H + \hbar)\eta^{n-1} = \cdots = \eta^n(H + n\hbar).$$

Let us now assume there is a ket $|S\rangle$ such that

$$\bar{\eta}|S\rangle = 0. \tag{4.2}$$

Then

$$H|S\rangle = (\eta\bar{\eta} + \tfrac{1}{2})\hbar|S\rangle = \tfrac{1}{2}\hbar|S\rangle$$

and $|S\rangle$ is an eigenket of H belonging to the eigenvalue $\frac{1}{2}\hbar$. Similarly,

$$H\eta|S\rangle = \eta(H + \hbar)|S\rangle = \tfrac{3}{2}\hbar\eta|S\rangle,$$
$$\vdots$$
$$H\eta^n|S\rangle = (n + \tfrac{1}{2})\hbar\eta^n|S\rangle.$$

Thus the kets $\eta^n|S\rangle$ are eigenkets of H belonging to the eigenvalues $(n + \frac{1}{2})\hbar$, for $n = 0, 1, 2, \ldots$.

One can easily see that if q and p are the only dynamical variables, there cannot be more than one independent ket satisfying (4.2). The way one would prove that would be to assume that there are two such kets, $|S_1\rangle$, $|S_2\rangle$, with $\bar{\eta}|S_1\rangle = 0$, $\bar{\eta}|S_2\rangle = 0$ and assume that they are independent; and then introduce a projection operator ω to satisfy

$$\omega\eta^n|S_1\rangle = \eta^n|S_1\rangle, \qquad \omega\eta^n|S_2\rangle = 0.$$

One easily checks that it is quite consistent to assume the existence of such a ω in the case when $|S_1\rangle$ and $|S_2\rangle$ are independent. One can also check that ω commutes with η and $\bar{\eta}$, and hence with q and p, so it is another q-number, another dynamical variable, independent of q and p. That would contradict our hypothesis that q and p are the only independent dynamical variables. So if q and p are the only independent dynamical variables, there is only one independent ket $|S\rangle$ satisfying (4.2).

We may write a general ket $|A\rangle$ in the form

$$|A\rangle = \sum_n a_n \eta^n |S\rangle = \psi(\eta)|S\rangle, \tag{4.3}$$

where $\psi(\eta)$ is a power series. We then have the function $\psi(\eta)$ to represent the ket $|A\rangle$. It is different from the usual kind of representation in that the variable η in the wave function is complex.

The conjugate bra to $|S\rangle$ is $\langle S|$, satisfying $\langle S|\eta = 0$. We assume $\langle S|S\rangle = 1$.

If we have two kets

$$|A\rangle = \sum_n a_n \eta^n |S\rangle \qquad \text{and} \qquad |B\rangle = \sum_n b_n \eta^n |S\rangle,$$

we have for the scalar product

$$\langle B|A\rangle = \sum_{nm} \langle S|\bar{b}_m \bar{\eta}^m a_n \eta^n |S\rangle.$$

We have

$$\begin{aligned}\langle S|\bar{\eta}^m \eta^n|S\rangle &= \langle S|\bar{\eta}^{m-1}(\eta^n \bar{\eta} + n\eta^{n-1})|S\rangle \\ &= n\langle S|\bar{\eta}^{m-1}\eta^{n-1}|S\rangle = n(n-1)\langle S|\bar{\eta}^{m-2}\eta^{n-2}|S\rangle = \cdots \\ &= \begin{cases} 0 & \text{for } m \neq n, \\ n! & \text{for } m = n, \end{cases}\end{aligned}$$

and hence $\langle B|A\rangle = \sum_n n!\bar{b}_n a_n$. If we normalize $\langle A|A\rangle = 1$, then $\sum_n n!|a_n|^2 = 1$ and $n!|a_n|^2$ is the probability of finding the system in the nth excited state for the state $|A\rangle$.

One can make an interesting development, as was done by Fock, by introducing η^{-1} to satisfy $\eta^{-1}\eta = 1$. We cannot have $\eta\eta^{-1} = 1$, because $\langle S|\eta = 0$, leading to $\langle S|\eta\eta^{-1} = 0$. We then find $n!\langle S|\eta^{-n} = \langle S|\bar{\eta}^n$, because, as is easily checked, both sides give the same result when multiplied on the left into $\eta^n|S\rangle$, or into $\eta^m|S\rangle$ with $m > n$, or into $\eta^m|S\rangle$ with $m < n$, provided we make the additional assumption $\eta^{-1}|S\rangle = 0$.

We now have

$$\langle B| = \langle S| \sum_n \bar{b}_n \bar{\eta}^n = \langle S| \sum n! \bar{b}_n \eta^{-n} = \langle S|\phi(\eta),$$

say, and $|B\rangle$ is thus represented by $\phi(\eta)$, a descending power series in η. The scalar product is now

$$\langle B|A\rangle = (2\pi i)^{-1} \oint \phi(\eta)\psi(\eta) \frac{d\eta}{\eta}$$

taken around the origin.

If there exist other dynamical variables besides p and q, then we may construct kets of the type (4.3), in which the coefficients a_n are not c-numbers but functions of the other dynamical variables. In that way we can get a more general set of kets built up from the one standard ket $|S\rangle$. For example, in the case I gave above, where there was just the projection operator ω, we can introduce dynamical variables which don't commute with ω and which enable one to pass from $|S_1\rangle$ to $|S_2\rangle$; and such dynamical variables could occur in the coefficients a_n and would lead to more general kets obtainable from $|S_1\rangle$ than the ones we should have if we took the a's to be c-numbers.

Those are the basic ideas of the Fock representation applied to one degree of freedom. The Fock representation is useful for any problem in which $\frac{1}{2}(p^2 + q^2)$ plays an important part. The Fock representation is always permissible, but it would only be useful if $\frac{1}{2}(p^2 + q^2)$ plays an important part in the calculations we are doing. A representation in quantum theory is like a system of coordinates in geometry; it is always permissible to take any system of coordinates that we like, but a system of coordinates is useful and convenient only when it is related in some special way to the things that are important for the problem under discussion. And here the Fock representation is useful when $\frac{1}{2}(p^2 + q^2)$ is important in the problem under discussion. Such problems are problems where we have an oscillator with energy $\frac{1}{2}(p^2 + q^2)$ and perhaps perturbations on this oscillator energy.

This theory can be extended to several oscillators, several q's and p's, where for each q and p we have some quantity $\frac{1}{2}(p^2 + q^2)$, perhaps with numerical coefficients coming in, playing an important part in our calculations. Then we introduce several η's, one referring to each p and q, so that we have a theory with the dynamical variables η_a, $\bar{\eta}_a$, the suffix a taking on different values to correspond to the different degrees of freedom. They will satisfy the conditions that the η's all commute with one another,

$$\eta_a \eta_b - \eta_b \eta_a = 0,$$

while

$$\bar{\eta}_a \eta_b - \eta_b \bar{\eta}_a = \delta_{ab}. \tag{4.4}$$

If we form $\eta_a \bar{\eta}_a$, we can call that n_a the degree of excitation of the nth oscillator. We define the standard ket $|S\rangle$ of the representation by the conditions

$$\bar{\eta}_a |S\rangle = 0 \qquad \text{for all} \quad \bar{\eta}_a. \tag{4.5}$$

Those conditions are consistent with one another because the $\bar{\eta}$'s commute with one another. As before, we can show that if the η's and $\bar{\eta}$'s are the only dynamical variables in our problem, there is only one independent $|S\rangle$ satisfying the conditions (4.5).

We can set up kets now of the form

$$|P\rangle = \eta_1{}^{n_1'} \eta_2{}^{n_2'} \cdots |S\rangle,$$

where n_1', n_2', . . . are c-numbers taking the values 0, 1, 2, 3, ... It is easily checked that $\eta_a \bar{\eta}_a |P\rangle = n_a' |P\rangle$, from the commutations relations between the η's and $\bar{\eta}$'s. That shows that $|P\rangle$ is an eigenket of each n_a belonging to the eigenvalue n_a'. We therefore get a ket that is an eigenket of the degrees of excitation of all the oscillators.

We may form a more general ket $|Q\rangle$ by superimposing all these:

$$|Q\rangle = \sum a_{n_1' n_2'} \cdots \eta_1{}^{n_1'} \eta_2{}^{n_2'} \cdots |S\rangle$$

summed over all the values of the n', where the coefficients are c-numbers.

§5. *Second Quantization: Bosons*

The Fock representation leads to the subject of second quantization, which forms a cornerstone in quantum field theory. I expect most of you know about it already, and to those I shall just say that you can keep all your previous knowledge on second quantization provided you apply it only to the relationships at one particular time t_0. I am going to restrict the discussion here to one particular time and to develop the theory of second quantization at one particular time.

Let us suppose that we have a particle that can exist in various states, at a certain time of course. The various states are described by eigenvalues for a complete set of commuting observables. Now, in talking about a particle we have only a finite number of degrees of freedom, and all the usual work about the equivalence of the Heisenberg and Schrödinger representations is then valid. In particular, all the independent states for one particle form a Hilbert space, so that in dealing with just one particle we have various states at a certain time corresponding to the vectors of a Hilbert space, and we can introduce coordinates for these vectors. We may use the usual notation in terms of the basic vectors $|a^1\rangle$, $|\alpha^2\rangle$, $|\alpha^3\rangle$, etc. We may take the alphas here to be a complete set of commuting observables for the one particle, and the suffixes 1, 2, 3, . . . refer to eigenvalues of those observables.

For an assembly of u such particles, we may have the state

$$|\alpha_1{}^a\rangle\,|\alpha_2{}^b\rangle \cdots |\alpha_u{}^g\rangle$$

obtained by just multiplying the kets for the individual particles. That is a possible state for the assembly. Such a state does not occur in nature; in nature we have all our particles satisfying either Bose statistics or Fermi statistics. If they satisfy Bose statistics, only symmetrical states occur in nature. If they satisfy Fermi statistics, only antisymmetrical states occur in nature.

Let's take the Bose case first. For bosons a possible state is the state that we get by symmetrizing the above ket between all the particles:

$$S|\alpha_1{}^a\rangle\,|\alpha_2{}^b\rangle \cdots |\alpha_u{}^g\rangle = S|\alpha_1{}^a\alpha_2{}^b \cdots \alpha_u{}^g\rangle, \tag{5.1}$$

where S is the symmetrizing operator, $S = (u!)^{-1/2}\sum P$, P being a permutation operator. The numerical coefficient $(u!)^{-1/2}$ is inserted for convenience of normalization. This is a possible state for an assembly of bosons. If the number of bosons is finite, the states for them form a Hilbert space.

One needs to check up on the normalization of the ket (5.1). One may assume that the original ket vectors are all orthonormal: $\langle\alpha^a|\alpha^b\rangle = \delta^{ab}$, so they form the base of a representation in Hilbert space. Under these conditions one finds that the squared length of $S|\alpha_1{}^a\alpha_2{}^b \cdots \alpha_u{}^g\rangle$ is 1 when the $a, b, \ldots, g$ states are all different, because all the terms in the sum (5.1) are then orthogonal to one another. If there are repetitions, some of them are not orthogonal. For example, if a and b are the same, the permutation that interchanges

a and b will give us a ket that is not orthogonal to the one we had before, but the same ket as before. The result is that we get a factor $n_1'!\, n_2'! \cdots$ in the squared length, where the n''s are the numbers of states that are equal. n_1' is the number of α's equal to α_1, etc.

Now let us suppose that the number of bosons u is variable and that u takes various eigenvalues u'. The totality of all the independent kets then consists of the following; first, there will be a ket corresponding to no bosons present, which we shall denote $|\ \rangle$; then there are kets with one boson present, $|\alpha_a\rangle$, kets with two bosons present, $S|\alpha^a\alpha^b\rangle$; kets with three bosons present, $S|\alpha^a\alpha^b\alpha^c\rangle$; and so on. We can now set up a whole series of kets describing various states for the assembly of bosons where the number of bosons present is allowed to vary.

A general ket is a sum of all these kets:

$$a|\ \rangle + a_a|\alpha^a\rangle + a_{ab}S|\alpha^a\alpha^b\rangle + \cdots,$$

where the coefficients are c-numbers and the coefficients involving more than one suffix are symmetrical.

Well, this is the way in which one describes the states for an assembly of bosons at a certain time. There is a connection between these kets and the kets we have for the set of oscillators described by the Fock representation. The connection is

$$S|\alpha^a\alpha^b \cdots \alpha^g\rangle = \eta_1^{n_1'}\eta_2^{n_2'} \cdots |S\rangle = \eta_a\eta_b \cdots \eta_g|S\rangle, \tag{5.2}$$

where the n''s are defined as before. (Don't be disturbed by having the same letter S for the symmetrizing operator and the standard ket.) We may put these kets equal because those on the left have the same properties as those on the right. The kets on the left are orthogonal to one another, just as the kets on the right. The squared length of each of the kets on the left is the same as the squared length of the corresponding ket on the right.

The fact that we have this connection means that the two dynamical systems are mathematically equivalent—one dynamical system being the set of bosons, the number of bosons being variable, and the other dynamical system being the set of oscillators. Those two dynamical systems are equivalent; they are just two ways of describing the same physical reality. It is this connection which establishes the equivalence between the wave and corpuscular theories of light. You may describe light either as waves, using the η variables, or as photons, using the kets $|\alpha\rangle$ for the various states of the photons.

I should like you to note that each oscillator corresponds not to a boson but to a state for a boson. There may be any number of bosons in a boson state, and their number n is the same as the degree of excitation of the corresponding oscillator. That is what we want physically. There may be any number of photons in a certain photon state; and the photon state corresponds to a Fourier component of the field, which is an oscillator. We have a boson state corresponding to an oscillator.

Now a question that suggests itself is the following: Suppose we started off the discussion of bosons referring to a different set of basic states for each boson. Above we had the set of states $|\alpha_1\rangle, |\alpha_2\rangle \cdots$ and so on. We might use a different set of states $|\beta_1\rangle, |\beta_2\rangle \cdots$ and so on, where the β's are another complete set of commuting observables for one particle. For example, the α's might be the position variables of the particles, and the β's might be the momentum variables. We can just as well formulate our discussion for the states of the assembly of bosons in terms of the β ket vectors instead of the α ket vectors. We have the α ket vectors satisfying the orthornormal conditions $\langle\alpha^b|\alpha^a\rangle = \delta^{ab}$, and we impose similar orthonormal conditions on these new ket vectors. I shall write a general one $|\beta^A\rangle$, because there is usually no one-to-one correspondence between the eigenvalues of the α's and the β's, and if we used the same index a in $|\beta\rangle$, it would imply such a one-to-one correspondence. Thus

$$\langle\beta^A|\beta^B\rangle = \delta^{AB}.$$

The kets for one particle are in Hilbert space; one particle is a system with just three degrees of freedom, or maybe an extra degree for the spin, and for such a simple particle the usual theory in terms of Hilbert vectors is quite correct. Both the α and the β vectors are in Hilbert space. The connection between them is given by the usual transformation of coordinates in Hilbert space; one of the basic ket vectors $|\beta\rangle$ is connected with the basic ket vectors $|\alpha\rangle$ by the equation

$$|\beta^A\rangle = |\alpha^a\rangle\langle\alpha^a|\beta^A\rangle, \tag{5.3}$$

with summation implied over all values of a.

We may work with these new kets to discuss an assembly of bosons, and this implies setting up the ket $S|\beta_1{}^A\beta_2{}^B \cdots \beta_u{}^G\rangle$ to describe u bosons. We may again introduce a set of oscillators described by a Fock representation, so that the states of these oscillators are connected with the states of the assembly of bosons by the same formula,

$$S|\beta_1{}^A\beta_2{}^B\cdots\beta_u{}^G\rangle = \eta_A\eta_B\cdots\eta_G|S\rangle.$$

The question arises: What is the connection between the new set of oscillators and the previous ones? There are these two sets of oscillators, both of them offering a description of the same assembly of bosons. Well, the ket $|S\rangle$ is the same in both cases; it refers to the state with no bosons present at all, and that, of course, doesn't make any reference to what kind of ket vectors we use to describe a boson. It is evident that the connections among the η's are linear ones, so that there is a homogeneous linear connection between the new η's and the old ones. Exactly what that connection is we can easily work out by considering the states of one boson. If we just have one boson, it may be described by the states $|\alpha^a\rangle = \eta_a|S\rangle$ and also by $|\beta^A\rangle = \eta_A|S\rangle$. Now we have the connection (5.3) between the $|\alpha^a\rangle|$ and $|\beta^A\rangle|$ and hence $\eta_A|S\rangle = \eta_a\langle\alpha^a|\beta^A\rangle|S\rangle$. Here $\langle\alpha^a|\beta^A\rangle$ is just a numerical factor, so we get $\eta_A = \eta_a\langle\alpha^a|\beta^A\rangle$. This formula shows us how the oscillator variables transform when we make a transformation of the basic kets to which we refer our bosons. One can sum up this result by saying that the η's transform in the same way as the basic kets for one boson.

When we describe one particle in the Schrödinger way we have a ket $|P\rangle$ to describe the state of one particle, which can be represented by a Schrödinger wave function, consisting of $\langle\alpha^a|P\rangle$, or $\langle\beta^A|P\rangle$. Now the wave function transforms like a basic bra; the transformation law is

$$\langle\beta^A|P\rangle = \langle\beta^A|\alpha^a\rangle\langle\alpha^a|P\rangle.$$

The transformation law for the wave function is therefore the same as the transformation law for the $\bar{\eta}$'s.

The η's have the physical meaning of being emission operators of a boson, or operators that increase the excitation of an oscillator by one stage; the $\bar{\eta}$'s are correspondingly absorption operators, or operators which reduce the degree of excitation of an oscillator by one stage. So we see that the wave function for one particle transforms like the absorption operators in the Fock representation.

This connection is known as *second quantization*. The reason for that name, of course, is that when you introduce the Schrödinger wave function you have performed a first quantization, if you start from the classical theory. Then second quantization consists in introducing operators $\bar{\eta}_a$ to correspond to the wave function $\langle\alpha^a|P\rangle$, and η_a to correspond to the conjugate complex wave function $\langle P|\alpha''\rangle$. Second quantization consists of passing from the c-numbers, which give us a wave function and its conjugate complex, describing one particle, to q-numbers that describe an assembly of bosons. The q-numbers differ from the c-numbers only in the fact that the c-numbers all commute with one another while the q-numbers don't, but satisfy $\bar{\eta}_a\eta_b - \eta_b\bar{\eta}_a = \delta_{ab}$ instead. So second quantization consists of introducing a lack of commutation into the quantities that describe one particle in the Schrödinger theory, and this lack of commutation then provides us with a set of oscillator variables suitable for describing an assembly of bosons.

I have written the formulas for the case when the basic kets for one particle are all discrete, but one can very well apply the same ideas to the case in which we have a continuous range of basic kets. The β's might very well be observables having a continuous range of eigenvalues. That, in fact, is usually the case; if the β's refer to momenta of a particle or position coordinates of a particle, they certainly have a continuous range of eigenvalues.

We have to make only a formal change in the equations when we have a continuous range of basic vectors. We have to replace the Kronecker δ by the δ-function; thus for continuous kets we have $\langle\beta^A|\beta^B\rangle = \delta(\beta^A - \beta^B)$. When we pass over to the corresponding oscillators referring to the Fock representation, we shall get η variables having suffixes that take on a continuous range of values, and those η variables will then satisfy the conditions

$$\begin{aligned}\bar{\eta}_A\eta_B - \eta_B\bar{\eta}_A &= \delta(\beta^A - \beta^B)\\ &= \delta(\beta_1{}' - \beta_1{}'')\delta(\beta_2{}' - \beta_2{}'')\delta(\beta_3{}' - \beta_3{}'')\cdots.\end{aligned}$$

§6. *Second Quantization: Fermions*

The above theory of second quantization can be applied to an assembly of bosons. There is a corresponding theory for an assembly of fermions. We may set it up in a way similar to the one we used for bosons. We start now with fermion particles, which can again be in various states; and we introduce, instead of the symmetrizing operator, an antisymmetrizing operator A defined by $A = (u!)^{-1/2}\sum \pm P$, where the sum is over all permutations, with a plus or minus sign in front of each, according to whether the permutation is even or odd.

We can form $A|\alpha_1{}^a\alpha_2{}^b \cdots \alpha_u{}^g\rangle$, which is antisymmetric between the various particles and which describes an assembly of fermions. It is now necessary that all the fermions be in different states; otherwise we get zero when we apply the antisymmetrizing operator. There is no longer any possibility of repetitions among the α's; the number of fermions in any fermion state must be 0 or 1.

In the case of bosons, the number of particles in any state is 0, 1, 2, 3, . . . , any integer up to infinity, and that suggests a connection with oscillators, because the energy values for one oscillator are an arithmetical progression. That is why the assembly of bosons is connected with a set of oscillators.

We want to do something corresponding to that for fermions, but it is certainly no good introducing oscillators, because the oscillators will have the wrong set of eigenvalues for their degrees of excitation. The number of particles in the various states are now variables, each of which has only two eigenvalues. That suggests variables like those which occur in the Pauli theory of spin one-half. We can develop the theory of fermions analogously to the case of bosons, by introducing operators η and $\bar{\eta}$, which are not connected with harmonic oscillators but are connected with variables similar to the spin variables that describe a half-quantum of spin.

Let's introduce some complex q-numbers η_a to satisfy

$$\begin{gathered}\eta_a\eta_b + \eta_b\eta_a = 0, \qquad \bar{\eta}_a\bar{\eta}_b + \bar{\eta}_b\bar{\eta}_a = 0, \\ \eta_a\bar{\eta}_b + \bar{\eta}_b\eta_a = \delta_{ab}.\end{gathered} \tag{6.1}$$

For $b = a$ these relations give

$$\eta_a{}^2 = 0, \tag{6.2}$$

$$\eta_a\bar{\eta}_a + \bar{\eta}_a\eta_a = 1. \tag{6.3}$$

Well, the first question that arises is: Are these relations consistent? Is it possible to introduce such η's? One can easily check their consistency in the following way. We can consider matrices whose squares are equal to unity and which anticommute with each other. We have three such in the Pauli spin theory for an electron, and we have a larger number when we go to the relativistic theory of the electron. By extending the number of rows and columns in our matrices we can obtain any number of these quantities, which all have their squares equal to unity and anticommute with one another. Let us suppose we have a large number of such quantities ξ, which all have their squares equal to 1 and anticommute; and let us label these ξ's, so that there are two of them corresponding to each value of a. Call them ξ_a and ξ_a'. Now take

$$\eta_a = \tfrac{1}{2}(\xi_a + i\xi_a'), \qquad \bar{\eta}_a = \tfrac{1}{2}(\xi_a - i\xi_a').$$

Then the η's and $\bar{\eta}$'s constructed in that way satisfy the required conditions, as is easily verified.

We can treat the η's and $\bar{\eta}$'s as we did in the Fock representation. We introduce the standard ket $|S\rangle$, satisfying

$$\bar{\eta}_a|S\rangle = 0 \qquad \text{for all } \bar{\eta}_a, \tag{6.4}$$

noting that these equations are consistent with one another, because we don't get into an inconsistency by forming

$$\begin{aligned} \bar{\eta}_a\bar{\eta}_b|S\rangle &= 0,\\ \bar{\eta}_b\bar{\eta}_a|S\rangle &= 0,\\ (\bar{\eta}_a\bar{\eta}_b + \bar{\eta}_b\bar{\eta}_a)|S\rangle &= 0. \end{aligned}$$

We can now set up more general kets $\eta_a\eta_b\eta_c \cdots |S\rangle$ and so on. All the η's here have to be different. If two of them were the same, for example, a and c, we could bring the η_c to the left of the factor η_b at the expense of a change of sign; and then we have the square of one η, which is zero from (6.2).

These kets can be connected with the kets for an assembly of fermions. We may put

$$A|\alpha_1{}^a\alpha_2{}^b \cdots \alpha_u{}^g\rangle = \eta_a\eta_b \cdots \eta_g|S\rangle. \tag{6.5}$$

The reason we can do that is similar to the reason it was permissible in the case of bosons. Two of the left-hand-side kets referring to different selections of states are orthogonal to each other. They are orthogonal to each other if they refer to different values of u; and if they refer to the same value of u they are also orthogonal to each other if they refer to different sets of states. In the same way, the kets on the right side are orthogonal to each other when they refer to different η factors. One can easily check that. I am going to check that $\eta_a\eta_b\eta_c \cdots \eta_g|S\rangle$ is orthogonal to $\eta_{a'}\eta_{b'} \cdots \eta_{g'}|S\rangle$, except when they have the same factors in front of $|S\rangle$. We form the scalar product of the conjugate of one of them with the other:

$$\langle S|\bar{\eta}_{g'} \cdots \bar{\eta}_{b'}\bar{\eta}_{a'}\eta_a\eta_b \cdots \eta_g|S\rangle.$$

We can change the order of the factors in accordance with the anticommutation relations (6.1). If we take any one of the $\bar{\eta}$'s, for instance, $\bar{\eta}_{g'}$, we can bring it over to the right and apply it directly to $|S\rangle$, and then it gives zero. The only condition under which we may have a nonzero result is when one of the η's is equal to $\eta_{g'}$. Similarly, for each of the other $\bar{\eta}_{i'}$ factors, there must be a corresponding η_i factor; otherwise the whole product is zero.

We have therefore on the right side of (6.5) a set of kets that are all orthogonal to each other when they refer to different sets of suffixes. If we take two of them referring to the same set of suffixes in a different order, then those two kets differ in sign or are equal to each other, according to whether the permutation that goes from one order to the other is odd or even. That is just the same situation that we have on the left side.

The kets on the left side of (6.5) are all normalized. In the boson case they were not all normalized, because when we have more than one boson in a certain state, that brings a factor not equal to unity into the square of the modulus of that ket. But in the case of the fermions we cannot have more than one particle in the same state, and that situation cannot arise. So the kets on the left of (6.5) are all normalized. The kets on the right of (6.5) are also normalized, provided, of course, we make the standard ket $|S\rangle$ normalized. That is easily checked in this way. We form $\langle S|\bar{\eta}_g \cdots \bar{\eta}_b\bar{\eta}_a\eta_a\eta_b \cdots \eta_g|S\rangle$. Now let us take these two middle factors $\bar{\eta}_a\eta_a$. From (6.3), we can replace them by $1 - \eta_a\bar{\eta}_a$. The factor $\bar{\eta}_a$ here can be shifted over to the extreme right, as there cannot be any other η_a, and $\bar{\eta}_a$ then gives zero by acting on $|S\rangle$. So the middle factors can be replaced by unity. Then, of course, we have $\bar{\eta}_b\eta_b$ as the two middle factors, and we treat them in the same way. They also give the factor unity, and so on. So the normalization is all right.

Now we have proved all that we need to establish that the two sets of kets in (6.5) may be identified with each other. We now have a way of describing a set of fermions in terms of these operators η and $\bar{\eta}$. We have the ket $|S\rangle$ corresponding to no fermions present; $\eta_a|S\rangle$ corresponds to one fermion in state a; $\eta_a\eta_b|S\rangle$ corresponds to two fermions in states a and b; and so on.

We have in this scheme that the number of fermions in any state is given by the operator $\eta_a\bar{\eta}_a = N_a$. The proof consists in applying N_a to each of the kets in the series above and seeing that we get the right result. $\bar{\eta}_a|S\rangle = 0$, so $|S\rangle$ is an eigenket of N_a belonging to the eigenvalue 0. $N_a\eta_a|S\rangle$ is equal to

$$\eta_a\bar{\eta}_a\eta_a|S\rangle = \eta_a(1 - \eta_a\bar{\eta}_a)|S\rangle = \eta_a|S\rangle,$$

so $\eta_a|S\rangle$ is an eigenket of N_a belonging to the eigenvalue 1. We check in the same way that $N_a\eta_b\eta_c \cdots |S\rangle$ equals 0 if $b, c, \ldots$ are all different from a; and $N_a\eta_a\eta_b \cdots |S\rangle = \eta_a\eta_b \cdots |S\rangle$. The factors $\eta_b\eta_c$ don't affect things here at all. That shows that for all the kets $\eta_b\eta_c \cdots |S\rangle$, N_a has the eigenvalue zero; and for all the kets $\eta_a\eta_b \cdots |S\rangle$, N_a has the eigenvalue 1. Thus N_a does represent the number of fermions in state a.

η_a can now be looked upon as the operator of creation of a fermion into the state a. If we apply η_a to any ket representing a state with no fermions present in state a, the resulting ket refers to a state with one fermion present in state a. The fact that we cannot put two fermions into the state is connected with ${\eta_a}^2 = 0$; if we apply the operator η_a twice we always get zero, no matter what we start with. Similarly, $\bar{\eta}_a$ is the operator of destruction of a fermion from state a: $\bar{\eta}_a$ gives zero unless there is a fermion previously present; and then it gives a new ket for which this fermion has been removed.

We can prove algebraically that the number of fermions in state a is 0 or 1. For that we form

$${N_a}^2 = \eta_a\bar{\eta}_a\eta_a\bar{\eta}_a = \eta_a(1 - \eta_a\bar{\eta}_a)\bar{\eta}_a = \eta_a\bar{\eta}_a - {\eta_a}^2{\bar{\eta}_a}^2 = \eta_a\bar{\eta}_a = N_a.$$

Thus ${N_a}^2 = N_a$; so the eigenvalues of N_a are 0 or 1.

We may make transformations to different states from our original fermion states. We can do that just as we did for bosons. That will give us a new set of kets for the assembly of fermions, which can be connected with new η and $\bar{\eta}$ variables. The connection between the old and new variables is similar to that for the case of bosons. The calculation goes exactly the same for fermions and bosons. Thus we have a theory of second quantization for fermions.

The final results for fermions and bosons can be connected in a single scheme of equations,

$$\eta_a\eta_b \pm \eta_b\eta_a = 0, \qquad \bar{\eta}_a\eta_b \pm \eta_b\bar{\eta}_a = \delta_{ab} \quad \text{or} \quad \delta(a - b),$$

where we have plus for fermions, minus for bosons. We have $\eta_a\bar{\eta}_a = N_a$, which holds for both fermions and bosons. Reversing the order of the factors, $\bar{\eta}_a\eta_a = 1 \pm N_a$, where the minus sign is for fermions and the plus sign for bosons.

Those are the important equations connecting the emission and absorption operators for fermions and bosons. It is really remarkable that there is such a similarity between the cases of fermions and bosons. Looked at from the particle point of view, they are very different; but expressed in terms of the operators η the equations become very similar in the two cases; there is just the difference in sign.

We can make one development of the theory of fermions to which there is nothing corresponding in the theory of bosons. Let us go back to the equations (6.1) for fermions. We note that there is symmetry between the η's and the $\bar{\eta}$'s. Put all the η's equal to $\bar{\eta}$'s, and all $\bar{\eta}$'s equal to η's, and the equations remain unchanged. In the case of the bosons that is not true, because the sign goes wrong.

Now let us see what this symmetry means physically. If we have a theory of an assembly of fermions, we can get another theory, in which we interchange the operators η and $\bar{\eta}$. Now

$$\eta_a\bar{\eta}_a = N_a, \qquad \bar{\eta}_a\eta_a = 1 - N_a.$$

So if we interchange η's and $\bar{\eta}$'s, N_a gets altered to $1 - N_a$. If the original N_a is equal to 1, the new N_a is zero and vice versa. It means that if the original state is occupied, the state in the new description is unoccupied, and vice versa. We get, by this transformation, a new description in which the concepts of full and empty states are interchanged.

The original description would be useful if most of the N's were 0 and there were just a few particles corresponding to a few of the N's being equal to 1. But if we have a situation in which most of the N's are equal to 1 and just a few of them equal to 0, we can make this transformation and get a new description in which most of the states are empty and just the few holes of the original description become the occupied states.

Of course, we don't have to apply this transformation to all the states. We might just apply it to some of them. For some values of a we might interchange η_a and $\bar{\eta}_a$, and not for others; and then the relations (6.1) are still all right. Thus just for some of the states we change the concepts of full and empty. Well, you know that that is what one does in the relativistic theory of electrons. Considering the electron as one particle by itself, studying its relativistic quantum theory, one finds that it has states of positive energy and states of negative energy. It becomes reasonable to interchange the concepts of full and empty states for the negative-energy states, so that we concentrate our attention on the originally unoccupied negative-energy states.

The ket $|S\rangle$ has to be changed when we make a transformation involving the interchange of η and $\bar{\eta}$. The original $|S\rangle$ satisfied the conditions $\bar{\eta}_a|S\rangle = 0$. When we interchange the η's and $\bar{\eta}$'s we are led to work with a new $|S\rangle$, satisfying $\eta_a|S_{\text{new}}\rangle = 0$. $|S_{\text{new}}\rangle$ is a ket for which, with the original picture of particles, we have all the states occupied. Multiplying it by $\bar{\eta}_a$ we get a state with one hole.

When we interchange η's and $\bar{\eta}$'s only for some degrees of freedom and not all, we need a ket $|S\rangle$ for which some $\bar{\eta}$'s applied to $|S\rangle$ are equal to 0 and some η's applied to $|S\rangle$ are equal to 0. The general $|S\rangle$ satisfies

$$\bar{\eta}_a|S_{\text{gen}}\rangle = 0 \qquad \text{for some } a\text{-values,}$$

$$\eta_b|S_{\text{gen}}\rangle = 0 \qquad \text{for } b\text{-values differing from the above } a\text{'s.}$$

When we are dealing with electrons and positrons it is convenient to set up such a ket for which the a-values include all the positive-energy states and the b-values include all the negative-energy states.

Quantum field theory is formulated entirely in terms of these operators η and $\bar{\eta}$, referring to either bosons or fermions. It is true quite generally that all quantum field theories are built up in terms of such η's and $\bar{\eta}$'s; they are the basic bricks one uses.

§7. *A Model Hamiltonian*

The η's and $\bar{\eta}$'s are the q-numbers that come into the description of nature at one particular time. We shall now discuss how they vary with time. We shall assume that there is a Hamiltonian and that we have Heisenberg's equations of motion giving us their variation with time.

As far as the conditions at one time are concerned, we have the kets $\eta_a\eta_b \cdots |S\rangle$, which lie in a Hilbert space if you impose suitable convergence conditions on the coefficients when you form sums

$$\sum_{abc\ldots} C_{abc\ldots}\eta_a\eta_b\eta_c \cdots |S\rangle. \tag{7.1}$$

One can in that way get an adequate description of the different states at one time, while sticking to one particular Hilbert space.

When we go over to conditions at another time, we would again have η's and $\bar{\eta}$'s referring to the new time, and we could again introduce $|S\rangle$ referring to the new time, and we would again get a set of Hilbert vectors. But these Hilbert vectors referring to the new time would not, in general, lie in the same Hilbert space as the set of ket vectors we started with. This is so for the Hamiltonians of interest in physics. The Hilbert space provided by the ket vectors changes, and if we want to consider the totality of ket vectors for all times we have to have a space larger than a Hilbert space.

This situation can best be discussed in terms of a model Hamiltonian. I propose to take a model Hamiltonian that is not one which occurs in practice at all, but is a good model because it is sufficiently simple for us to be able to calculate explicitly everything that we want to calculate and yet is sufficiently complicated to show up the need for passing from one Hilbert space to another and the impossibility of representing everything in terms of one Hilbert space and using the Schrödinger picture.

We shall work with fermions, so we have a set of dynamical variables η_m, $\bar{\eta}_m$ satisfying (6.1); we shall take $m = 1, 2, \ldots, \infty$. It is necessary to have an infinite number of fermion states, as the difficulties do not arise when there is only a finite number.

We take as the Hamiltonian,

$$H = \tfrac{1}{2}(a_{mn}\eta_m\eta_n + \bar{a}_{mn}\bar{\eta}_n\bar{\eta}_m), \tag{7.2}$$

where the coefficients a are c-numbers. This Hamiltonian is real. It just involves quadratic terms in the emission and absorption operators.

Now, because $\eta_m\eta_n = -\eta_n\eta_m$, any symmetrical matrix a will give zero; only the skew part of this matrix contributes anything, so we take $a_{mn} = -a_{nm}$. We then have a Hamiltonian that we may also write

$$H = \tfrac{1}{2}(a_{mn}\eta_m\eta_n - \bar{a}_{mn}\bar{\eta}_m\bar{\eta}_n).$$

Now let us introduce a ket vector $|S\rangle$ to satisfy $\bar{\eta}_m|S\rangle = 0$ for all m. This ket vector corresponds to no fermions present.

Let us now form $H|S\rangle$. It is, of course, $\tfrac{1}{2}a_{mn}\eta_m\eta_n|S\rangle$, the second term in H contributing nothing. This $H|S\rangle$ just consists of terms each referring to two fermions being present.

Now let us form $H^2|S\rangle$. We get terms referring to the existence of four fermions and terms with no fermions at all. The terms containing four fermions present are pretty straightforward and there is no problem with them. I want to concentrate on those terms which contain no fermions present at all. They come from

$$-\tfrac{1}{4}\bar{a}_{mn}\bar{\eta}_m\bar{\eta}_n a_{pq}\eta_p\eta_q|S\rangle.$$

Using the anticommutation relations (6.1), we find that this is equal to

$$\begin{aligned}-\tfrac{1}{4}\bar{a}_{mn}a_{pq}\bar{\eta}_m(\delta_{np} - \eta_p\bar{\eta}_n)\eta_q|S\rangle \\ &= -\tfrac{1}{4}[\bar{a}_{mn}a_{nq}\bar{\eta}_m\eta_q - \bar{a}_{mn}a_{pq}\bar{\eta}_m\eta_p(\delta_{nq} - \eta_q\bar{\eta}_n)]|S\rangle \\ &= -\tfrac{1}{2}\bar{a}_{mn}a_{nq}\bar{\eta}_m\eta_q|S\rangle = -\tfrac{1}{2}\bar{a}_{mn}a_{nq}(\delta_{mq} - \eta_q\bar{\eta}_m)|S\rangle \\ &= -\tfrac{1}{2}\bar{a}_{mn}a_{nm}|S\rangle = -\tfrac{1}{2}\langle\bar{a}a\rangle|S\rangle, \end{aligned} \tag{7.3}$$

where $\langle\quad\rangle$ indicates the diagonal sum of the enclosed matrix. The matrix $\bar{a}$ is defined by $(\bar{a})_{mn} = \overline{a_{mn}}$. It is not the conjugate Hermitian.

Now we may very well have the diagonal sum infinite even with a perfectly respectable matrix a, a bounded matrix, for example. We may have

$$\bar{a}a = -1, \tag{7.4}$$

for example. This could be achieved by taking as the nonzero element

$$a = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix}. \tag{7.5}$$

This is a matrix satisfying the antisymmetry condition and is a very simple and well-behaved matrix, and yet we get an infinity in (7.3) and we have to say that $H^2|S\rangle$ does not exist.

Suppose we try to get a solution of the Schrödinger equation; i.e., we try to solve

$$i\hbar\frac{d}{dt}|A\rangle = H|A\rangle, \tag{7.6}$$

starting off with $|A\rangle = |S\rangle$ at $t = 0$. A formal solution of this equation is $|A\rangle = e^{-iHt/\hbar}|S\rangle$, because, looking at things from the abstract point of view and assuming that one can differentiate the exponential according to the ordinary rule for differentiating an exponential that contains a numerical parameter, this would just satisfy (7.6), and it also satisfies the right initial condition. But, you see, if you try to give a meaning to the exponential as a power series, the terms of the second degree in t already fail, as they involve the second degree in H and $H^2|S\rangle$ does not exist.

You might wonder whether $e^{-iHt/\hbar}|S\rangle$ does have a meaning, in spite of its being impossible to expand the exponential. Maybe the exponential applied to the ket does exist, even though H^2 applied to the ket does not exist. Before giving a definite answer to this question, I should like to discuss the problem from the Heisenberg point of view.

Take this same Hamiltonian and apply it in the Heisenberg picture. The Heisenberg equations of motion give

$$\begin{aligned} i\hbar\frac{d\eta_p}{dt} = \eta_p H - H\eta_p &= -\tfrac{1}{2}\bar{a}_{mn}(\eta_p\bar{\eta}_m\bar{\eta}_n - \bar{\eta}_m\bar{\eta}_n\eta_p) \\ &= -\tfrac{1}{2}\bar{a}_{mn}\{(\delta_{pm} - \bar{\eta}_m\eta_p)\bar{\eta}_n - \bar{\eta}_m(\delta_{pn} - \eta_p\bar{\eta}_n)\} \\ &= -\tfrac{1}{2}(\bar{a}_{pn}\bar{\eta}_n - \bar{a}_{mp}\bar{\eta}_m) = -\bar{a}_{pn}\bar{\eta}_n\,. \end{aligned}$$

Then, of course, we have the conjugate equation

$$-i\hbar\frac{d}{dt}\bar{\eta}_p = -a_{pn}\eta_n\,.$$

Well, those Heisenberg equations of motion are very simple. The η's and $\bar{\eta}$'s all vary linearly in terms of themselves. The equations can very easily be integrated.

Writing the equations in matrix form, we have

$$\hbar\frac{d\eta}{dt} = i\bar{a}\bar{\eta}, \qquad \hbar\frac{d\bar{\eta}}{dt} = -ia\eta,$$

and then

$$\hbar^2\frac{d^2\eta}{dt^2} = aa\eta\,.$$

We see that the general solution is of the type

$$\eta(t) = A\eta(0) + B\bar{\eta}(0),$$

where A and B are matrices whose elements are c-numbers and functions of t. Then one gets

$$\hbar \frac{dA}{dt} = i\bar{a}\overline{B}, \qquad \hbar \frac{dB}{dt} = i\bar{a}\overline{A}, \qquad \hbar^2 \frac{d^2 A}{dt^2} = \bar{a}aA.$$

Putting it like this, we get equations in terms of matrices instead of q-numbers. q-numbers are rather mysterious things, and we like to get them out of the calculation when we can and replace them by matrices involving c-numbers as their elements, which are quite well understood.

Now let's take the example (7.4). Then the elements of A and B just vary according to the law of $\cos(t/\hbar)$ and $\sin(t/\hbar)$. We can easily fit them in to have the right initial conditions, at $t = 0$:

$$A = 1, \qquad B = 0, \qquad \frac{dA}{dt} = 0, \qquad \frac{dB}{dt} = i\bar{a}/\hbar,$$

and the solution is

$$\eta_r(t) = \cos(t/\hbar)\eta_r(0) + i\sin(t/\hbar)\bar{a}_{rs}\bar{\eta}_s(0). \tag{7.7}$$

There you see we have integrated the Heisenberg equations of motion, and we ran into no difficulty. The kind of infinity we had earlier is a difficulty peculiar to the Schrödinger picture, which one avoids by working with the Heisenberg picture. That kind of difficulty is typical of quantum field theory. In quantum field theory we deal, of course, with substantially more complicated Hamiltonians than this model I've dealt with, but we have the same general features that the Heisenberg equations of motion are easier to work with than the Schrödinger ones. We may very well be concerned in practice with an example for which there are infinities even with the Heisenberg picture. I am not saying that the Heisenberg picture eliminates all the infinities, but it eliminates some that occur in the Schrödinger picture and is superior to the Schrödinger picture in that way.

Now let us go back to the question I left unanswered: Does the ket $e^{-iHt/\hbar}|S\rangle$ exist? Looking at things from the Heisenberg point of view, we have our η's varying with the time in accordance with (7.7). In the Heisenberg picture we may introduce a ket $|S_0\rangle$ to satisfy $\bar{\eta}_n(0)|S_0\rangle = 0$. It is reasonable to assume that there is such a ket. If we *can* introduce kets in the Heisenberg picture, that is presumably one of them, about the simplest one that one can think of. It is the same as our previous Schrödinger ket $|S\rangle$. But we may equally well introduce $|S_t\rangle$ to satisfy $\bar{\eta}_m(t)|S_t\rangle = 0$. One time is as good as another in the Heisenberg picture and if $|S_0\rangle$ exists, $|S_t\rangle$ should exist also.

What is the relationship between those two kets? Well, that is easily worked out. The formal solution of the Heisenberg equation is

$$\bar{\eta}_m(t) = e^{iHt/\hbar}\bar{\eta}_m(0)e^{-iHt/\hbar}.$$

Let us substitute this expression for $\bar{\eta}_m$ in the equation $\bar{\eta}_m(t)|S_t\rangle = 0$. We get

$$e^{iHt/\hbar}\bar{\eta}_m(0)e^{-iHt/\hbar}|S_t\rangle = 0 \qquad \text{or} \qquad \bar{\eta}_m(0)e^{-iHt/\hbar}|S_t\rangle = 0.$$

Let us compare this equation with the equation $\bar{\eta}_m(0)|S_0\rangle = 0$. We see that

$$e^{-iHt/\hbar}|S_t\rangle = |S_0\rangle,$$

apart from a numerical factor, a c-number factor.

So $|S_t\rangle = e^{iHt/\hbar}|S_0\rangle$, and it would seem that $e^{iHt/\hbar}|S_0\rangle$ exists. The fact that we cannot expand the exponential means that the ket is not in the Hilbert space of kets (7.1) with $|S_0\rangle$ for $|S\rangle$. Here we have a ket that does not lie in that Hilbert space; apparently it does not lie in any Hilbert space at all. Thus we cannot get a solution of the Schrödinger equation if we keep to Hilbert space. We can get a solution if we are willing to accept kets in a more general space than a Hilbert space. But these more general kets cannot be described by coordinates. As far as is known at present, there is no way of introducing coordinates to describe the vectors in this more general ket space. The Schrödinger wave function consists of the coordinates of the ket and therefore the Schrödinger wave function does not exist. So we get the situation in which there is an abstract solution of the Schrödinger equation, but there is no Schrödinger wave function.

Suppose we started off with a different ket, not this $|S_0\rangle$. Could we perhaps have better luck with the Schrödinger picture? We might try to start off with a ket for which the Hamiltonian is diagonal. Well let's see what happens then. I shall take the example (7.4) for the matrix a. It is good enough to illustrate the points we want to discuss.

Let us go back to the solution (7.7) of the Heisenberg equations. The conjugate solution is

$$\bar{\eta}(t) = \cos(t/\hbar)\bar{\eta}(0) - i\sin(t/\hbar)a\eta(0).$$

Suppose we form

$$\eta(t) + \bar{a}\bar{\eta}(t) = e^{it/\hbar}[\eta(0) + \bar{a}\bar{\eta}(0)].$$

We may write it out explicitly with the matrix a given by (7.5),

$$\begin{aligned}
\eta_1(t) + \bar{\eta}_2(t) &= e^{it/\hbar}[\eta_1(0) + \bar{\eta}_2(0)],\\
\eta_2(t) - \bar{\eta}_1(t) &= e^{it/\hbar}[\eta_2(0) - \bar{\eta}_1(0)],\\
\eta_3(t) + \bar{\eta}_4(t) &= e^{it/\hbar}[\eta_3(0) + \bar{\eta}_4(0)],\\
\eta_4(t) - \bar{\eta}_3(t) &= e^{it/\hbar}[\eta_4(0) - \bar{\eta}_3(0)],
\end{aligned}$$

and so on. This way of writing things shows how the variables connect up to give quantities that vary with time according to the law $e^{it/\hbar}$, that is, quantities that just get multiplied by a numerical factor in the Heisenberg picture as times goes on.

We might now set up a ket vector $|S^*\rangle$ in the Heisenberg picture satisfying

$$\begin{aligned}
(\eta_1 + \bar{\eta}_2)|S^*\rangle &= 0, \qquad (\eta_2 - \bar{\eta}_1)|S^*\rangle = 0,\\
(\eta_3 + \bar{\eta}_4)|S^*\rangle &= 0, \qquad (\eta_4 - \bar{\eta}_3)|S^*\rangle = 0,
\end{aligned} \tag{7.8}$$

and so on. These equations remain valid for all time with the same $|S^*\rangle$, because the operators on $|S^*\rangle$ just get multiplied by numerical factors as time goes on. We should check, of course, that the equations (7.8) are consistent. It won't do setting up equations for a ket which are not consistent with one another. Let us take the first two and let us form the anticommutator[†]

$$[\eta_1 + \bar{\eta}_2, \eta_2 - \bar{\eta}_1]_+ = -[\eta_1, \bar{\eta}_1]_+ + [\bar{\eta}_2, \eta_2]_+ = 0.$$

The consistency of the others is checked in the same way. All these equations are therefore consistent and (7.8) is just as good a way of defining a ket as any of the previous ones.

So there we have defined a ket $|S^*\rangle$. Now how would that ket behave in the Schrödinger picture. Well, we must express the Hamiltonian in terms of the variables occurring in (7.8) and their conjugates. We can do that in this way: Using the given value (7.5) for the matrix a,

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[†] We use the notation,

$$\begin{aligned}
[u, v]_+ &= uv + vu,\\
[u, v]_- &= uv - va,\\
[u, v] &= (uv - vu)/i\hbar.
\end{aligned}$$

H will contain as the first two terms

$$\eta_1\eta_2 - \bar{\eta}_1\bar{\eta}_2.$$

We can express them in the following way:

$$\begin{aligned}\eta_1\eta_2 - \bar{\eta}_1\bar{\eta}_2 &= \tfrac{1}{2}[(\eta_1 - \bar{\eta}_2)(\eta_2 - \bar{\eta}_1) - (\eta_2 + \bar{\eta}_1)(\eta_1 + \bar{\eta}_2) \\ &\qquad + \eta_1\bar{\eta}_1 + \bar{\eta}_2\eta_2 + \eta_2\bar{\eta}_2 + \bar{\eta}_1\eta_1] \\ &= \tfrac{1}{2}[(\eta_1 - \bar{\eta}_2)(\eta_2 - \bar{\eta}_1) - (\eta_2 + \bar{\eta}_1)(\eta_1 + \bar{\eta}_2)] + 1.\end{aligned}$$

Let us now replace H by another Hamiltonian,

$$\begin{aligned}H_1 = \tfrac{1}{2}[(\eta_1 - \bar{\eta}_2)(\eta_2 - \bar{\eta}_1) - (\eta_2 + \bar{\eta}_1)(\eta_1 + \bar{\eta}_2)]& \\ + \tfrac{1}{2}(\text{same with } \eta_3\eta_4) + \cdots.&\end{aligned}$$

H_1 differs from H by a c-number. The c-number has a contribution 1 from each pair of fermion states, so the total difference is an infinite c-number.

From the point of view of the Heisenberg picture, H_1 is equivalent to H; they both give just the same Heisenberg equations of motion. From the point of view of the Schrödinger equation, H_1 and H are different. Now, $H_1|S^*\rangle = 0$, so $|S^*\rangle$ is a solution of the Schrödinger equation for H_1. If we try to work with H we should have $|S^*\rangle$ changing by a phase factor and changing infinitely rapidly. If you want to interpret the Hamiltonian as the total energy, you would have to say that H_1 and H correspond to two different physical systems in which the total energy differs merely by an infinite constant. From the physical point of view, H_1 is at least as reasonable to work with as H, and if we work with H_1 then we do have a solution of the Schrödinger equation. Not merely a formal solution but a solution that stays in Hilbert space; in fact, it doesn't move at all.

Well, then the statement, "One can't get a solution to the Schrödinger equation in Hilbert space" really needs modification. One should put it this way. One cannot get a solution approximating to $|S_0\rangle$. One can get a solution differing very substantially from $|S_0\rangle$ if one suitably defines the zero-point energy.

That is the mathematical situation. The question is: What are the important features that arise when we want to apply this theory to the Hamiltonians that occur in practice? We have our Hamiltonian provided for us and we want to get solutions that are physically useful. Solutions that are physically useful will be those close to the vacuum state, with just a few particles present, if any. Solutions close to the vacuum state correspond to kets close to $|S_0\rangle$, for which one fails to get a solution to the Schrödinger equation in Hilbert space. If you take quantum electrodynamics with the Hamiltonian I shall obtain later, you could get solutions corresponding to $|S^*\rangle$, but that kind of solution would not be of any help to the physicist, because it would differ too much from the physical vacuum.

I am really anticipating future lectures when I talk about the Hamiltonian for quantum electrodynamics, but I would like you to have a clear picture of what the situation is. You know that in applying the second quantization theory to electrons you have to suppose that the physical vacuum is the state for which all the positive-energy electron states are unoccupied and all the negative-energy states are occupied. Such a state is like the $|S_0\rangle$ that occurs above. You would get a state like $|S^*\rangle$ if you supposed that all the electron states are unoccupied, both the positive-energy and negative-energy ones. You might start off with the mathematical situation in which all the electron states are unoccupied to begin with, and you could treat the ket corresponding to that state as we did $|S^*\rangle$. This $|S^*\rangle$ didn't change at all in the Schrödinger picture. But going over to the correct Hamiltonian of electrodynamics, we get an $|S^*\rangle$ that changes a little but not seriously. However, that ket is not one a physicist can use, because it differs too drastically from the kets that describe physical situations. An example of such a ket for electrodynamics is given in my paper, *Comm. Dublin Inst. Adv. Studies, Ser. A* (No. 3) (1946), Sec. 3.

To say that the Schrödinger equation has no solutions is putting things too drastically. In the first place, it has symbolic solutions that don't stay in Hilbert space; second, it does have some solutions that stay in Hilbert space but differ too much from what the physicist wants to be of any use. Under those conditions I would maintain that the Schrödinger picture is unworkable and the Heisenberg picture is the one we have to restrict ourselves to.

I've given this discussion in terms of a very primitive Hamiltonian. One could develop the formalism to work with rather more complicated Hamiltonians which illustrate the same features. There is a more powerful formalism given in a paper of mine (The Vacuum in Quantum Electrodynamics) in *Nuovo Cimento, Suppl.* **6**, p. 322 (1957). In Section 4 of this paper I prove a number of theorems that are useful for getting kets which don't lie in a Hilbert space. I shall just enumerate the theorems. I shall use ψ's instead of η's to agree with the notation of the paper, so we have

$$\psi_A\psi_B + \psi_B\psi_A = 0, \qquad \psi_A\bar{\psi}_B + \bar{\psi}_B\psi_A = \delta_{AB}.$$

In this work ψ will be considered as a column matrix and $\bar{\psi}$ as a row matrix. Let us introduce a square matrix λ whose matrix elements are c-numbers.

One of the theorems is

$$[\psi, \bar{\psi}\lambda\psi]_- = \lambda\psi. \tag{7.9}$$

Here $\bar{\psi}\lambda\psi$ is a single element, because λ is a square matrix, and when it is multiplied by a column matrix on the right and a row matrix on the left, the result is a single element. The right side of (7.9) is a column matrix, because we have here a column matrix ψ multiplied on the right into the square matrix λ. One checks the theorem by using the anticommutation relations, quite a straightforward piece of work. There is the conjugate theorem to (7.9), which is

$$[\bar{\psi}, \bar{\psi}\lambda\psi]_- = -\bar{\psi}\lambda. \tag{7.10}$$

When you have proved those two you can easily go on to prove

$$[\bar{\psi}\mu\psi, \bar{\psi}\lambda\psi]_- = \bar{\psi}[\mu, \lambda]_-\psi, \tag{7.11}$$

where μ is another square matrix with c-number elements, like λ. One can go on from that to prove that

$$e^{\bar{\psi}\lambda\psi}\psi e^{-\bar{\psi}\lambda\psi} = e^{-\lambda}\psi. \tag{7.12}$$

Here $\bar{\psi}\lambda\psi$ is a single element, and we can form its exponential in the usual way. There is a conjugate theorem

$$e^{\bar{\psi}\lambda\psi}\bar{\psi}e^{-\bar{\psi}\lambda\psi} = \bar{\psi}e^{\lambda}. \tag{7.13}$$

Those are the theorems that are useful for the present kind of work.

Suppose a ket $|X\rangle$ satisfies

$$(a\psi + \bar{\psi}b)|X\rangle = 0, \tag{7.14}$$

where a is a row matrix of c-numbers and b is a column matrix of c-numbers. Define

$$|Y\rangle = e^{\bar{\psi}\lambda\psi}|X\rangle. \tag{7.15}$$

Then from (7.12) and (7.13),

$$(ae^{-\lambda}\psi + \bar{\psi}e^{\lambda}b)|Y\rangle = 0. \tag{7.16}$$

It takes a whole lot of equations (7.14) to define the ket $|X\rangle$. These equations might be, for example, that all the absorption operators for a positive-energy electron applied to $|X\rangle$ give zero and all the emission operators for a negative-energy electron give zero. Then we are able to construct another ket $|Y\rangle$ from it, satisfying a different set of linear conditions (7.16). These conditions (7.16) would have to be consistent if the conditions (7.14) satisfied by $|X\rangle$ are consistent. That is, if the operators $a\psi + \bar{\psi}b$ all anticommute with each other, the operators $ae^{-\lambda}\psi + \bar{\psi}e^{\lambda}b$ must necessarily all anticommute with each other also. Thus we get a set of consistent conditions to specify the new ket $|Y\rangle$.

It might very well be that if we try to expand the exponential in (7.15) we get infinities coming in, just as with the more primitive example we were considering earlier. Under those conditions the ket $|Y\rangle$ would not be in the same Hilbert space as the ket $|X\rangle$. That is a fairly general way of constructing kets that lie outside Hilbert space.

When I was working on this some years ago, I was hoping to be able to get a ket lying outside the usual Hilbert space but suitable to define the physical vacuum at all times, and so satisfy the Schrödinger equation. That proved to be impossible. One can get the ket to satisfy the conditions up to a certain degree of accuracy, but going to greater accuracy the conditions fail. It has led me to believe that it is impossible to get a ket to describe the physical vacuum at all times.

§8. *Value of the Classical Theory*

We must now face the problem of finding the right Hamiltonian to be used in our quantum theory to get a description of nature. For this we work from a classical analog. This is simply because it would be too difficult to think of a quantum theory straight away. We want to have something to help us. The only thing that we can have to help us is a classical model.

We must put the classical theory in a form suitable for passing to the quantum theory; the form that we need is the Hamiltonian form. Now you might say: Can we not immediately start off with the Hamiltonian form of classical theory and work from that to a quantum theory? That would not be convenient, because we want our theory to be relativistic, and for the Hamiltonian form of classical mechanics it is not very easy to see whether the theory is relativistic or not.

It is best to start from a classical action principle. It is very easy to see whether an action principle is relativistic or not: If the action quantity is Lorentz-invariant, everything has to be relativistic. So we see that it is best to start off from an action principle.

We then make an assumption of localization: that each bit of action is localized in time. This is sufficient to provide a Lagrangian, which is just the action per unit time. For a relativistic theory, we should also require each bit of action to be localized in space, so that it becomes localized in space-time.

There are standard methods for passing from the Lagrangian to the Hamiltonian. These methods are discussed in my book *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monograph Series No. 2.

Let us consider now the connection between classical theory and quantum theory; and let us take the example of a classical theory which, when expressed in Hamiltonian form, does not involve any constraints. I want to leave the question of how to handle constraints until later and for the present just consider a classical Hamiltonian theory without constraints. To quantize this theory we take the dynamical variables q and p occurring in the classical Hamiltonian theory and make them into q-numbers. We have a Hamiltonian function in the classical theory, and in simple examples we can assume that we have the same Hamiltonian in quantum theory. The Hamiltonian of the classical theory is a certain function of the classical dynamical variables and we set up the quantum Hamiltonian which is the same function of the quantum dynamical variables; but I would like you to note that this procedure of quantization is not uniquely determined. Given a classical Hamiltonian, we cannot in general say with certainty what the corresponding quantum Hamiltonian is. The classical Hamiltonian may involve a product of two factors that don't commute; now, when we go over to the quantum theory, what are we to do about this product? The classical theory doesn't tell us the order in which to put the two factors; we have to decide on an order when we go over to the quantum theory.

It is, of course, necessary that the quantum Hamiltonian be real, as it is the total energy of the system; that's something physicists can measure, and has to be a real thing. The quantum Hamiltonian has to be a real q-number, but this condition of reality is not in general sufficient to fix which quantum Hamiltonian corresponds to a particular classical Hamiltonian. Given one real quantum Hamiltonian, we can add to it $\hbar$ times any real quantity, and get another quantum Hamiltonian which corresponds to the same classical one.

The result of this is that when we've got a given classical theory, in general there is not a unique quantum theory corresponding to it. There is no well-defined unique process for passing from classical theory to quantum theory. That means that when we set up a quantum theory we have to set it up to stand on its own feet, independent of the classical theory. The only value of the classical theory is to provide us with hints for getting a quantum theory; the quantum theory is then something that has to stand in its own right. If we were sufficiently clever to be able to think of a good quantum theory straight away, we could manage without classical theory at all. But we're not that clever, and we have to get all the hints that we can to help us in setting up a good quantum theory.

The classical theory can help us quite a lot. It provides us with suitable Hamiltonians with which we can start to work in the quantum theory. We can study them and see if they are good or not. When you've got one of the Hamiltonians and you've studied it, you might find that you have to modify it; but still you've got a start. Without the classical theory you wouldn't have this start at all.

I think we may ultimately reach the stage when it is possible to set up quantum theory without any reference to classical theory, just as we already have reached the stage where we can set up the Einstein gravitational theory without any reference to the Newtonian theory. But from the point of view of teaching students, I think one would always have to proceed by stages—not expect too much from them, teach them first the elementary theories and gradually develop their minds; and that will always involve working from the classical theory first.

Well, as far as constructing the Hamiltonian goes, it is just a question of hints. But there is a transformation theory of classical mechanics which is closely connected with the transformation theory of quantum mechanics; that connection arises from the similarity of the Poisson brackets in the two cases. The similarity of the Poisson brackets is something stronger than hints; it's a very close connection between classical theory and quantum theory. You can't modify the definition of the quantum Poisson bracket; that is uniquely determined by analogy to the classical Poisson bracket.

§9. *The Scalar Field*

As a simple example of a field that may be useful for the description of nature, we consider a real scalar field V, associated with particles of zero rest mass.

The coordinates of space-time will be denoted $x_\mu (\mu = 0, 1, 2, 3)$. We raise and lower suffixes according to the rules: For any contravariant vector A_μ, we take $A_0 = A^0$, $A_r = -A^r (r = 1, 2, 3)$. The derivative of a scalar function of position such as V will be written $\partial V/\partial x_\mu = V^{,\mu}$. People who work in general relativity usually write the coordinates of space-time as x^μ, which would make all the suffixes opposite to the present notation.

Beginning with the classical theory, we take for the action density

$$\mathcal{L} = (8\pi)^{-1} V^{,\mu} V_{,\mu}. \tag{9.1}$$

It leads to the field equation

$$V^{,\mu}{}_{,\mu} = 0 \qquad \text{or} \qquad \Box V = 0, \tag{9.2}$$

where $\Box = \partial^2/\partial x_\mu \, \partial x^\mu$.

The Lagrangian is

$$\begin{aligned} L = \int \mathcal{L} \, d^3x &= (8\pi)^{-1} \int V_{,\mu} V^{,\mu} \, d^3x \\ &= (8\pi)^{-1} \int (\dot{V}^2 - V^{,r} V^{,r}) \, d^3x. \end{aligned}$$

This equals kinetic energy — potential energy, as is usual in particle dynamics.

The dynamical coordinates are the values of V at all points of three-dimensional space at a certain time. We may write them V_x, where the suffix x denotes x_1, x_2, x_3. The derivatives in space-like directions $V^{,r}$ are functions of the coordinates. The velocities are $\dot{V}_x = V^{,0}$.

The momentum U_x is the functional derivative $\delta L/\delta \dot{V}_x$. To define this functional derivative we vary the velocities in L and pick out the coefficient of $\delta \dot{V}_x$. We have $\delta L = (4\pi)^{-1} \int \dot{V} \delta \dot{V} d^3x$, and hence $U = \dot{V}/4\pi$. It is more convenient to redefine U equal to $\dot{V}$, so that it is 4π times the momentum conjugate to V.

With the new definition of U, the classical Hamiltonian for the scalar field is

$$H = (4\pi)^{-1} \int U\dot{V} \, d^3x - L = (8\pi)^{-1} \int (U^2 + V^{,r} V^{,r}) \, d^3x. \tag{9.3}$$

Note that H is positive-definite.

We have the Poisson-bracket relations

$$\begin{aligned} [V_x, U_{x'}] &= 4\pi \, \delta(x_1 - x_1') \, \delta(x_2 - x_2') \, \delta(x_3 - x_3') \\ &= 4\pi \, \delta_3(x - x'), \\ [V_x, V_{x'}] &= 0, \qquad [U_x, U_{x'}] = 0. \end{aligned}$$

They are the natural generalization of the Poisson-bracket relations for particles, applied to the infinite number of degrees of freedom of the field.

The Hamiltonian equations of motion are

$$\dot{V}_x = [V_x, H] = [V_x, (8\pi)^{-1} \int U_{x'}{}^2 \, d^3x'] = U_x,$$

$$\dot{U}_x = [U_x, H] = [U_x, (8\pi)^{-1} \int V_{x'}{}^{,r'} V_{x'}{}^{,r'} \, d^3x'] = V^{,rr},$$

which reproduce the field equations (9.2).

We shall now make a Fourier resolution of the field. Let $V(x) = V(x_0, x_1, x_2, x_3)$, to be distinguished from $V_x = V(x_1 x_2 x_3)$, in which a particular x_0 is understood. The field equation (9.2) shows that we can put

$$V(x) = \int \left(V_k{}^c e^{ik\cdot x} + \bar{V}_k{}^c e^{-ik\cdot x} \right) d^3k,$$

where

$$k \cdot x = k_0 x_0 - k_1 x_1 - k_2 x_2 - k_3 x_3 = k_0 x_0 - (k, x)$$

and

$$k_0 = (k_1{}^2 + k_2{}^2 + k_3{}^2)^{1/2} = |k| > 0,$$

and

$$d^3k = dk_1 \, dk_2 \, dk_3 .$$

Here $V_k{}^c$ means $V^c_{k_1 k_2 k_3}$ and is a constant Fourier coefficient.

For a given x_0,

$$V_x = \int (V_k e^{-i(\mathbf{k},\mathbf{x})} + \bar{V}_k e^{i(\mathbf{k},\mathbf{x})}) \, d^3k = \int (V_k + \bar{V}_{-k}) e^{-i(\mathbf{k},\mathbf{x})} \, d^3k, \tag{9.4}$$

where $V_k = V_k{}^c e^{ik_0 x_0}$ is a time-dependent Fourier coefficient. Also

$$U_x = \left(\frac{\partial V}{\partial x_0} \right)_{x_1 x_2 x_3} = i \int |k| (V_k - \bar{V}_{-k}) e^{-i(\mathbf{k},\mathbf{x})} \, d^3k. \tag{9.5}$$

We can invert these equations and get

$$\begin{aligned} V_k + \bar{V}_{-k} &= (2\pi)^{-3} \int V_x e^{i(\mathbf{k},\mathbf{x})} \, d^3x, \\ i|\mathbf{k}|(V_k - \bar{V}_{-k}) &= (2\pi)^{-3} \int U_x e^{i(\mathbf{k},\mathbf{x})} \, d^3x. \end{aligned} \tag{9.6}$$

V_x and $U_x = \partial V_x/\partial x_0$ at a given x_0 fix V_k for this x_0. The important thing is that the equations connecting V_x, U_x with V_k, $\overline{V}_{-k}$ do not involve x_0. So we can consider V_k, $\overline{V}_{-k}$ as new dynamical variables and they will satisfy $\dot{V}_k = [V_k, H]$ and not

$$\dot{V}_k = (\partial V_k/\partial t) + [V_k, H].$$

The relationship of the V_k and $V_k{}^c$ should be clearly understood. The V_k are the important quantities, of the nature of Hamiltonian dynamical variables. They and their conjugate complexes $\overline{V}_k$ at any time x_0 are defined in terms of the field quantities at x_0 by equations that do not involve x_0. If the field V is interacting with other things, one may still use the V_k, $\overline{V}_k$ as Hamiltonian dynamical variables. The $V_k{}^c$, on the other hand, are integrals of the equations of motion, not working dynamical variables. They are integrals only for the case when the field V is not interacting with other things. They cease to be well defined if the field is interacting with other things.

Next we get the Hamiltonian in terms of the new variables V_k, $\overline{V}_k$. We have

$$\begin{aligned}\int U^2\, d^3x &= -\iiint |k|\,|k'|(V_k - \overline{V}_{-k})e^{-i(\mathbf{k},\mathbf{x})} \\ &\qquad \times (V_{k'} - \overline{V}_{-k'})e^{-i(\mathbf{k}',\mathbf{x})}\, d^3k\, d^3k'\, d^3x \\ &= -8\pi^3 \iint |k|\,|k'|(V_k - \overline{V}_{-k})(V_{k'} - \overline{V}_{-k'}) \\ &\qquad \times \delta_3(k + k')\, d^3k\, d^3k' \\ &= -8\pi^3 \int |k|^2(V_k - \overline{V}_{-k})(V_{-k} - \overline{V}_k)\, d^2k,\end{aligned}$$

$$\begin{aligned}\int V^{,r^2}\, d^3x &= -\iiint k_r(V_k + \overline{V}_{-k})e^{-i(\mathbf{k},\mathbf{x})}k_r{}'(V_{k'} + \overline{V}_{-k'}) \\ &\qquad \times e^{-i(\mathbf{k}',\mathbf{x})}\, d^3k\, d^3k'\, d^3x \\ &= 8\pi^3 \int k^2(V_k + \overline{V}_{-k})(V_{-k} + \overline{V}_k)\, d^3k,\end{aligned}$$

and

$$H = (8\pi)^{-1} \int (U^2 + V^{,r^2})\, d^3x = 2\pi^2 \int k^2(V_k\overline{V}_k + \overline{V}_{-k}V_{-k})\, d^3k \tag{9.7}$$

or

$$H = 4\pi^2 \int k^2 V_k \overline{V}_k\, d^3k. \tag{9.8}$$

For the Poisson-bracket relations, we have from (9.6),

$$[V_k + \bar{V}_{-k}, V_{k'} + \bar{V}_{-k'}] = 0 \tag{9.9}$$

$$[V_k - \bar{V}_{-k}, V_{k'} - \bar{V}_{-k'}] = 0 \tag{9.10}$$

$$\begin{aligned} i|k'|[V_k + \bar{V}_{-k}, V_{k'} - \bar{V}_{-k'}] &= (2\pi)^{-6} \iint [V_x e^{i(\mathbf{k},\mathbf{x})}, U_{x'} e^{i(\mathbf{k}',\mathbf{x}')}] \\ &\quad \times d^3x\, d^3x' \\ &= (16\pi^5)^{-1} \iint e^{i(\mathbf{k},\mathbf{x})} e^{i(\mathbf{k}',\mathbf{x}')}\, \delta_3(x - x') \\ &\quad \times d^3x\, d^3x' \\ &= (16\pi^5)^{-1} \int e^{i(\mathbf{k}+\mathbf{k}',\mathbf{x})}\, d^3x \\ &= (2\pi^2)^{-1}\, \delta_3(k + k'). \end{aligned} \tag{9.11}$$

From (9.10) and (9.11) we find

$$[V_k, V_{k'} - \bar{V}_{-k'}] = \frac{-i}{4\pi^2|k|}\, \delta_3(k + k').$$

From (9.9) and (9.11) with k and k' interchanged we have

$$[V_k, V_{k'} + \bar{V}_{-k'}] = \frac{i}{4\pi^2|k|}\, \delta_3(k + k').$$

Thus finally

$$[V_k, V_{k'}] = 0 \tag{9.12}$$

$$[V_k, \bar{V}_{-k'}] = \frac{i}{4\pi^2|k|}\, \delta_3(k + k')$$

or

$$[V_k, \bar{V}_{k'}] = \frac{i}{4\pi^2|k|}\, \delta_3(k - k'). \tag{9.13}$$

We may check the equation of motion:

$$\dot{V}_k = [V_k, H] = 4\pi^2 k^2 V_k \frac{i}{4\pi^2|k|} = i|k|V_k,$$

which agrees with the definition

$$V_k = V_k^c e^{i|k|x_0}.$$

We may quantize the theory according to Bose statistics. This involves replacing (9.12) and (9.13) by

$$V_k V_{k'} - V_{k'} V_k = 0 \tag{9.14}$$

and

$$V_k \overline{V}_{k'} - \overline{V}_{k'} V_k = \frac{-h}{4\pi^2 |k|} \delta_3(k - k'). \tag{9.15}$$

We compare these expressions with the ones coming from the Fock representation for a set of oscillators,

$$\eta_r \eta_s - \eta_s \eta_r = 0, \qquad \overline{\eta}_r \eta_s - \eta_s \overline{\eta}_r = \delta(r - s).$$

We see that V_k corresponds to an emission operator

$$V_k = (2\pi)^{-1} \left(\frac{h}{|k|} \right)^{1/2} \eta_k$$

and $\overline{V}_k$ to an absorption operator

$$\overline{V}_k = (2\pi)^{-1} \left(\frac{h}{|k|} \right)^{1/2} \overline{\eta}_k .$$

We cannot have V_k corresponding to the absorption operator and $\overline{V}_k$ to the emission operator, because the right side of (9.15) is a negative quantity when not zero.

We have in the quantum theory, as in the classical theory,

$$H = (8\pi)^{-1} \int (U^2 + V^{,r^2})\, d^3x = 2\pi^2 \int k^2 (V_k \overline{V}_k + \overline{V}_k V_k)\, d^3k.$$

from (9.7). In the quantum theory this equals

$$4\pi^2 \int k^2 V_k \overline{V}_k \, d^3k + \text{infinite } c\text{-number}. \tag{9.16}$$

In the Heisenberg equations of motion we may use for the Hamiltonian either one of the two expressions

$$H = (8\pi)^{-1} \int (U^2 + V^{,r^2})\, d^3x \qquad \text{or} \qquad 4\pi^2 \int k^2 V_k \overline{V}_k \, d^3k,$$

but whereas the first equation gives an infinite contribution to the vacuum energy, the second applied to the vacuum state $|0\rangle$ satisfying $\overline{V}_{k|0\rangle=0}$ gives zero and is thus the physical energy.

If we quantize according to Fermi statistics we put

$$V_kV_{k'} + V_{k'}V_k = 0, \qquad V_k\overline{V}_{k'} + \overline{V}_{k'}V_k = \frac{h}{4\pi|k|}\,\delta_3(k - k').$$

The expression $H = (8\pi)^{-1} \int (U^2 + V_{,r}{}^2)\, d^3x$ for the Hamiltonian is no longer any good, because it does not give the correct variation of V_k with time. It is then not possible to localize the energy. For this reason the quantization of the scalar field according to Fermi statistics is not of interest.

§10. *The Scalar Field: Relativistic Invariance*

The question arises whether the quantum theory of the scalar field with Bose statistics is really relativistic. The classical theory is certainly relativistic.

There are new assumptions made when one goes over from classical theory to quantum theory. One cannot be sure that the new assumptions are relativistic, because they refer to Hamiltonian variables, and all the Hamiltonian variables are defined in a nonrelativistic way. They're defined as referring to one particular time. We have to check whether our quantum theory is really relativistic. We use our classical theory only to suggest the quantum theory, and then we have to check whether that quantum theory is satisfactory by making calculations entirely in the quantum theory.

In the present case we have to check whether our commutation relations are independent of the Lorentz frame of reference with which we are working. We can do that in a direct manner by working out the commutator of V at two different points in space-time and seeing whether the result is relativistic. For the present let us take the case of the scalar field by itself. If there were other fields interacting with it, the calculation would have to be on different lines.

We have

$$[V_k, \overline{V}_{k'}] = i(4\pi^2|k|)^{-1}\,\delta_3(k - k').$$

For our present calculation it is convenient to work with the constant Fourier coefficients and not the time-dependent ones; they are related by $V_k{}^c = V_k e^{-i|k|x_0}$, and, of course, $\overline{V}_k{}^c = V_k e^{i|k|x_0}$. Let us work out the Poisson bracket of two of these constant Fourier coefficients:

$$[V_k{}^c, \bar{V}_{k'}{}^c] = [V_k, \bar{V}_{k'}]e^{-i|k|x_0}e^{i|k'|x_0} = i(4\pi^2|k|)^{-1}\,\delta_3(k - k').$$

The result is that we get here just the same formula for the constant Fourier coefficients as for the time-dependent ones. And of course, we have $[V_k{}^c, V_{k'}{}^c] = 0$. Noncommutation appears only when we have one V and one $\bar{V}$.

Let us now take

$$[V(x), V(x')] = \iint [V_k{}^c e^{ik\cdot x} + \bar{V}_k{}^c e^{-ik\cdot x}, V_{k'}{}^c e^{ik'\cdot x'} + \bar{V}_{k'}{}^c e^{-ik'\cdot x'}] \\ \times d^3k\, d^3k'.$$

Here $k \cdot x$ is the four-dimensional scalar product. We get

$$\begin{aligned}[V(x), V(x')] &= \frac{i}{4\pi^2}\iint \frac{1}{|k|} \\ &\quad \times (e^{ik\cdot x}e^{-ik'\cdot x'} - e^{-ik\cdot x}e^{ik'\cdot x'})\,\delta^3(k - k')\,d^3k\,d^3k' \\ &= \frac{i}{4\pi^2}\int |k|^{-1}[e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}]\,d^3k.\end{aligned}$$

Here k_0 means $|k|$. Now the convenient way of working this out is to rewrite it

$$\frac{i}{4\pi^2}\int |k|^{-1}\{e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}\}\,\delta(k_0 - |k|)\,d^4k,$$

where k_0 is now an independent variable. Now let us change the sign of all the k's in the second term; this implies giving negative values to k_0, but there's no harm in that, because k_0 is now an independent variable. So we get

$$\frac{i}{4\pi^2}\int |k|^{-1}e^{ik\cdot(x-x')}[\delta(k_0 - |k|) - \delta(k_0 + |k|)]\,d^4k \cdots. \qquad (10.1)$$

That's the result and we want to see whether it is Lorentz-invariant.

Suppose we have any four-vector with components a_0, a_1, a_2, a_3. Then $\delta(a_\mu a^\mu)$ is Lorentz-invariant. The function $\delta(a^\mu a_\mu)$ splits off into two parts, one referring to positive a_0 and one referring to negative a_0; and we may change the sign of that part for which a_0 is negative and still get a Lorentz-invariant function, $(a_0/|a_0|)\delta(a^\mu a_\mu)$. It doesn't really look Lorentz-invariant because you've got a_0 appearing in it, which refers to one particular Lorentz frame, but the Lorentz frame we take would not affect things, because $a_0/|a_0|$ is just a plus or minus sign, the same for any Lorentz frame of reference, because we don't consider time reversal.

We define the function $\Delta(a)$ as equal to $(2a_0/|a_0|)\,\delta(a_\mu a^\mu)$. This Δ-function is quite an important function in the field theory of particles with zero rest mass and it is a Lorentz-invariant function. Let us study it a bit. We can write $a^\mu a_\mu = a_0{}^2 - |a|^2 = (a_0 - |a|)(a_0 + |a|)$. Now whenever we have a quantity like $\delta(xy)$, it is equal to

$$|x|^{-1}\,\delta(y) + |y|^{-1}\,\delta(x).$$

That's a general formula for the theory of distributions and we very easily check it by inserting both members as a factor in an integral and integrating things out.

Let us use that result in our definition for Δ. We get

$$\Delta(a) = \frac{2a_0}{|a_0|}[|a_0 + |a||^{-1}\,\delta(a_0 - |a|) + |a_0 - |a|\,|^{-1}\,\delta(a_0 + |a|)].$$

Now in the first term we can put a_0 equal to $|a|$ and get $2|a|$ in the denominator; so the first term becomes $|a|^{-1}\,\delta(a_0 - |a|)$. Similarly the second term is $-|a|^{-1}\,\delta(a_0 + |a|)$ and we have

$$\Delta(a) = (|a|)^{-1}\,[\delta(a_0 - |a|) - \delta(a_0 + |a|)]. \tag{10.2}$$

Expression (10.1) now becomes

$$\frac{i}{4\pi^2}\int e^{ik\cdot(x-x')}\,\Delta(k)\,d^4k. \tag{10.3}$$

That proves that it is a Lorentz-invariant function.

The function Δ is interesting and is worth further study. I'd like to mention some of its properties. If we make a Fourier resolution, what do we get? I need not go through the analysis. The result is

$$\int \Delta(k)e^{ik\cdot a}\,d^4k = 4\pi^2 i\,\Delta(a). \tag{10.4}$$

We get the same function again. We may interchange a and x here and get

$$\int \Delta(a)e^{ik\cdot a}\,d^4a = 4\pi^2 i\,\Delta(x). \tag{10.5}$$

So $\Delta(k)$ is a function that transforms into itself under Fourier resolution, apart from a numerical coefficient.

We can use that result in expression (10.3) and we get

$$[V(x), V(x')] = -\Delta(x - x'). \tag{10.6}$$

That is the final result. It means that the field function V at two different points in space-time commute except when the two points lie on the light cones of each other. When that is the case, we have a sort of δ-function singularity.

I'd like to mention some other properties of the Δ-function. We have $\Delta(-a) = -\Delta(a)$: It's an odd function; it changes sign when you change the sign of all four a's. If $a_0 = 0$, $\Delta(a) = 0$. It's not clear from the definition just what $\Delta(a)$ is at the origin, but when you take into account its antisymmetric property you see that it has to vanish at the origin and everywhere on the three-dimensional surface $a_0 = 0$.

Another property of the Δ-function that I want to deduce is the value of its derivative at $a_0 = 0$. Let us evaluate $(\partial/\partial a_0)\,\Delta(a)$ and then put $a_0 = 0$. We can get the result most directly by using the Fourier resolution (10.4),

$$\begin{aligned}\frac{\partial\Delta(a)}{\partial a_0} &= \frac{1}{4\pi^2}\int k_0 e^{ik\cdot a}\,\Delta(k)\,d^4k\\ &= \frac{1}{4\pi^2}\int \frac{k_0}{|k|}[\delta(k_0 - |k|) - \delta(k_0 + |k|)]e^{ik\cdot a}\,d^4k\\ &= \frac{1}{4\pi^2}\int [\delta(k_0 - |\mathbf{k}|) + \delta(k_0 + |\mathbf{k}|)]e^{ik\cdot a}\,d^4k.\end{aligned}$$

Now let us put $a_0 = 0$; we get

$$\left.\frac{\partial\Delta(a)}{\partial a_0}\right|_{a_0=0} = \frac{1}{4\pi^2}\int [\delta(k_0 - |\mathbf{k}|) + \delta(k_0 + |\mathbf{k}|)]e^{-i(k,a)}\,d^4k.$$

(k, x) is only a three-dimensional scalar product. k_0 now no longer occurs in the integrand except in the δ-functions, and therefore the integration with respect to k_0 is trivial. The result is

$$\left.\frac{\partial}{\partial a_0}\Delta(a)\right|_{a_0=0} = 4\pi\,\delta_3(a).$$

We can now understand the nature of the singularity in the Δ-function at the origin. The function itself vanishes at the origin and its derivative with respect to a_0 is something which has the ordinary three-dimensional δ-function value when we consider it as spread over a_1, a_2, a_3.

I would like you to note that

$$\frac{\partial^2}{\partial a_\mu \partial a^\mu} \Delta(a) = 0. \tag{10.7}$$

This follows at once from the Fourier resolution.

If we look upon the a's as coordinates of space, this Δ-function consists entirely of waves traveling with the velocity of light. These waves form a sort of pulse, an ingoing pulse to begin with, a pulse that goes to the origin, goes through the origin, and comes out again still as a pulse. That is how the Δ-function is to be pictured. Just at the origin, the function itself vanishes but its time derivative does not vanish. Those are the important properties of the Δ-function.

You see why it comes into field theory in the equation for the commutator $[V(x), V(x')]$. You obviously need to have for this commutator something that can be resolved into Fourier components traveling entirely with the velocity of light, because $(\partial^2/\partial x_\mu\, \partial x^\mu)\, V(x) = 0$, and if you apply the operator $\partial^2/\partial x^\mu\, \partial x_\mu$ to $[V(x), V(x')]$, the result is

$$\frac{\partial^2}{\partial x_\mu\, \partial x^\mu}[V(x), V(x')] = \left[\frac{\partial^2}{\partial x_\mu\, \partial x^\mu} V(x), V(x')\right] = 0.$$

This checks with

$$\frac{\partial^2}{\partial x_\mu\, \partial x^\mu} \Delta(x - x') = 0.$$

That completes the proof that this scalar field theory is a relativistic theory even in the quantum theory. All our commutation relations are Lorentz-invariant. One could do a corresponding calculation for a nonzero rest mass. The results are essentially the same. We no longer have the Δ-function occurring; we have another expression involving a Bessel function, but it can likewise be proved to be Lorentz-invariant.

§11. *The Electron Field*

With the electron field we have four complex field functions ψ_α $(\alpha = 1, 2, 3, 4)$. They may conveniently be considered as the four elements of a column matrix ψ. They are assumed to satisfy the field equation

$$i\hbar(\psi'^0 + \alpha_r\psi'^r) - m\alpha_m\psi = 0, \tag{11.1}$$

where the α's are 4 × 4 Hermitian matrices multiplied into the column matrix ψ by matrix multiplication. Any two of them anticommute and the square of each is unity. The field equations (11.1) are relativistic when one makes the four components of ψ transform correctly under Lorentz transformations. They have to transform according to the laws of a spinor.

The conjugate complex field quantities $\bar{\psi}_a$ are considered as a row matrix $\bar{\psi}$. Thus the matrix product $\bar{\psi}\alpha\psi$, for any of the α matrices, is a single element. The conjugate complex equation to (11.1) is

$$i\hbar(\bar{\psi}'^0 + \bar{\psi}'^r\alpha_r) + m\bar{\psi}\alpha_m = 0. \tag{11.2}$$

Let us first consider the field classically. Because the field equations are first degree in $\partial/\partial x_0$, it is very easy to put them in Hamiltonian form. One finds that one gets the correct field equations (11.1) and (11.2) if one takes

$$H = \int (-i\hbar\bar{\psi}_x\alpha_r\psi_x'^r + m\bar{\psi}_x\alpha_m\psi_x)\, d^3x, \tag{11.3}$$

with the Poisson brackets

$$[\psi_{ax}, \psi_{bx'}] = 0, \qquad i\hbar[\psi_{ax}, \bar{\psi}_{bx'}] = \delta_{ab}\,\delta_3(x - x'). \tag{11.4}$$

Here ψ_{ax} is to be understood as ψ_a at the point x_1, x_2, x_3, for a particular time x_0 which is understood, like the notation V_x in Section 9. The $\hbar$ occurring in (11.3) and (11.4) is just a parameter in the classical theory.

It should be noted that the Hamiltonian (11.3) is real, because if one takes the conjugate complex integrand in (11.3) and makes an integration by parts, one gets back to the original expression.

We have here the classical theory, and to apply quantization to it it is convenient to introduce Fourier components. In the case of the scalar field we made the Fourier resolution first four-dimensionally, and then we passed over from the four-dimensional to the three-dimensional Fourier resolution. Here we can make the three-dimensional Fourier resolution directly. We put

$$\psi_x = h^{-3/2}\int e^{i(\mathbf{x},\mathbf{p})/\hbar}\psi_p\, d^3p,$$

with the conjugate equation·

$$\psi_p = h^{-3/2}\int e^{-i(\mathbf{x},\mathbf{p})/\hbar}\psi_x \, d^3x.$$

This applies for each of the four components. ψ_p has four components like ψ_x and is also understood as a column matrix.

Now let us express the Hamiltonian in terms of the Fourier components. We get

$$\begin{aligned} H &= h^{-3/2}\iint \bar{\psi}_x(\alpha_r p_r + \alpha_m m)e^{i(\mathbf{x},\mathbf{p})/\hbar}\psi_p \, d^3x \, d^3p \\ &= \int \bar{\psi}_p(\alpha_r p_r + \alpha_m m)\psi_p \, d^3p. \end{aligned} \tag{11.5}$$

We see that $\alpha_r p_r + \alpha_m m$ is the energy operator associated with each Fourier component p.

Now there are four Fourier components ψ_p. $\alpha_r p_r + \alpha_m m$ has positive eigenvalues for two of them and negative eigenvalues for the other two. There's a mixing up of the positive- and negative-energy Fourier components, and to carry on we have to separate them out and treat them differently. To do that we must transform the matrix $\alpha_r p_r + \alpha_m m$ into the diagonal form. This can be done by a suitable unitary transformation.

We choose U_p so that $U_p\bar{U}_p = 1$ and such that $U_p(\alpha_r p_r + \alpha_m m)\bar{U}_p$ is a diagonal matrix. The diagonal elements are plus or minus the square root of $p^2 + m^2$ because $(\alpha_r p_r + \alpha_m m)^2 = p^2 + m^2$, so $(\alpha_r p_r + \alpha_m m)$ has the eigenvalues $\pm(p^2 + m^2)^{1/2}$. So the diagonal form of the matrix $\alpha_r p_r + \alpha_m m$ is

$$\begin{pmatrix} (p^2 + m^2)^{1/2} & 0 & 0 & 0 \\ 0 & (p^2 + m^2)^{1/2} & 0 & 0 \\ 0 & 0 & -(p^2 + m^2)^{1/2} & 0 \\ 0 & 0 & 0 & -(p^2 + m^2)^{1/2} \end{pmatrix}.$$

Let us suppose we have found this unitary matrix U. There are many possible such matrices; it's not uniquely determined at all, but any one of them will do for our purposes. Now we can write

$$U_p(\alpha_r p_r + \alpha_m m)\bar{U}_p = (p^2 + m^2)^{1/2}V, \tag{11.6}$$

where V is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We now express H in the form

$$H = \int \bar{\psi}_p \bar{U}_p U_p (\alpha_r p_r + \alpha_m m) \bar{U}_p U_p \psi_p \, d^3p.$$

I've merely introduced $\bar{U}_p U_p$, which is equal to 1. We get

$$H = \int \bar{\psi}_p \bar{U}_p (p^2 + m^2)^{1/2} V U_p \psi_p \, d^3p.$$

Now let us put $U_p \psi_p = \phi_p$. ϕ_p also has four components. We get

$$H = \int (p^2 + m^2)^{1/2} \bar{\phi}_p V \phi_p \, d^3p.$$

There we've done the transformation which separates out the positive- and negative-energy parts of the field function ψ. If we write H out in full it reads

$$H = \int (p^2 + m^2)^{1/2} (\bar{\phi}_{1p}\phi_{1p} + \bar{\phi}_{2p}\phi_{2p} - \bar{\phi}_{3p}\phi_{3p} - \bar{\phi}_{4p}\phi_{4p}) \, d^3p. \tag{11.7}$$

We might quantize this H according to Bose quantization; the result would be a theory in which some of the particles have negative energy. The Hamiltonian here would contain some negative-energy terms, which would result in a nonphysical theory with negative-energy particles, which we don't want. Alternatively we might use Bose quantization and change the Hamiltonian, putting plus signs in the third and fourth terms of (11.7). That would give us a theory with only positive-energy particles, but then we should have destroyed the localized expression for the Hamiltonian.

We cannot apply Bose quantization to this theory without either bringing in negative-energy particles or spoiling localization. We can, however, apply Fermi quantization while keeping the localization and keeping only positive-energy particles. That is why we use Fermi quantization for this kind of field.

With Fermi quantization we may take

$$H = \int (p^2 + m^2)^{1/2}(\bar{\phi}_{1p}\phi_{1p} + \bar{\phi}_{2p}\phi_{2p} + \phi_{3p}\bar{\phi}_{3p} + \phi_{4p}\bar{\phi}_{4p})\, d^3p. \tag{11.8}$$

This differs from (11.7) by an infinite constant. We have interchanged the order of ϕ_{3p} and $\bar{\phi}_{3p}$ and of ϕ_{4p} and $\bar{\phi}_{4p}$, which involves bringing in a minus sign with Fermi quantization and also involves a constant to be added on. So this new Hamiltonian is equal to the preceding one apart from an infinite *c*-number. An infinite *c*-number, of course, doesn't invalidate the use of this Hamiltonian in the Heisenberg equations of motion. It will only come into play when we try to interpret the Hamiltonian as an energy.

Now if we're going to use (11.8) as our Hamiltonian and are going to have only positive-energy particles, each of the operators that occurs to the right of a product must be an absorption operator in order that it give zero when applied to the vacuum ket. We then have a vacuum ket $|0\rangle$ with $H|0\rangle = 0$ by taking $\phi_{1p}|0\rangle = \phi_{2p}|0\rangle = \bar{\phi}_{3p}|0\rangle = \bar{\phi}_{4p}|0\rangle = 0$. That's how we have to define our vacuum ket in this case. It's the only way to do it if we are to have all our particles with a positive energy. It involves counting ϕ_{1p}, ϕ_{2p}, $\bar{\phi}_{3p}$, $\bar{\phi}_{4p}$ as absorption operators and $\bar{\phi}_{1p}$, $\bar{\phi}_{2p}$, ϕ_{3p}, ϕ_{4p} as emission operators. As far as the ϕ_1 and ϕ_2 components of ϕ are concerned, we just have the usual kind of second quantization referring to the emission and absorption of the positive-energy electrons, which appear as ordinary electrons. As far as the 3 and the 4 components are concerned, we have interchanged the operations of emission and absorption that a direct quantization procedure would give us. This means we are working with a picture where the real physical things are just holes in what we would get with a direct method of second quantization. These holes are the positrons.

We have to interpret ϕ_{3p} and ϕ_{4p} as emission of a positron with momentum $-p$. In the primitive method of quantizing, where ϕ_{3p} and ϕ_{4p} are absorption operators of electrons in negative-energy states, these operators imply the absorption of a quantity of momentum p, and if we are going to reinterpret them as emission operators they have to be the emission operators of a particle with momentum $-p$.

Of course, ϕ_{1p} and ϕ_{2p} are the emission operators of an electron with momentum $+p$.

The classical Poisson bracket relations (11.4) get replaced by the following with Fermi quantization:

$$[\phi_{ap}, \phi_{bp'}]_+ = \phi_{ap}\phi_{bp'} + \phi_{bp'}\phi_{ap} = 0$$

and

$$[\phi_{ap}, \bar{\phi}_{bp'}]_+ = \delta_{ab}\, \delta_3(p - p').$$

These are the basic anticommutation relations coming into the Fermi quantization of the electron field.

How does it go if we transform back to the ψ's? Let's work that out. We have evidently $[\psi_{ap}, \psi_{bp'}]_+ = 0$. Further,

$$[\psi_{ap}, \bar{\psi}_{bp'}]_+ = [U_{pac}\phi_{cp}, \bar{\phi}_{dp'}\bar{U}_{p'db}]_+ .$$

The matrix elements of U_p and $\bar{U}_p$ are just numbers. They can be taken out of the anticommutator, so that we get

$$\begin{aligned}[\psi_{ap}, \psi_{bp'}]_+ &= \bar{U}_{pac}U_{p'db}\, \delta_{cd}\, \delta_3(p - p') \\ &= \delta_{ab}\, \delta_3(p - p'),\end{aligned}$$

where we have used

$$\bar{U}_{pac}U_{pcb} = \delta_{ab}.$$

So the anticommutation relations referred to the ψ's are just the same as referred to the ϕ's. This kind of unitary transformation doesn't disturb the anticommutation relations.

We can now transform a stage further and go back to the original ψ_x's, and we find

$$[\psi_{ax}, \psi_{bx'}]_+ = 0 \qquad \text{and} \qquad [\psi_{ax}, \bar{\psi}_{bx'}]_+ = \delta_{ab}\, \delta_3(x - x').$$

The Fourier transformation back from the p's to the x's doesn't affect the anticommutation relations.

The Hamiltonian is the same as the original Hamiltonian (11.3) which we had in the classical theory, except for an infinite constant. Of course, this infinite constant just represents the energy of the sea of negative-energy electrons.

I want now to get a formula for the total charge. The charge is made up from a contribution of $-e$ for each electron and $+e$ for each positron:

$$
\begin{aligned}
Q &= -e\int[\bar\phi_{1p}\phi_{1p} + \bar\phi_{2p}\phi_{2p} - \phi_{3p}\bar\phi_{3p} - \phi_{4p}\bar\phi_{4p}]\,d^3p \\
&= -e\int[\bar\phi_{1p}\phi_{1p} + \bar\phi_{2p}\phi_{2p} + \bar\phi_{3p}\phi_{3p} + \bar\phi_{4p}\phi_{4p}]\,d^3p + \text{inf. const.}
\end{aligned}
$$

This infinite constant corresponds to the total charge of all the negative-energy electrons. Now the above expression, neglecting the infinite constant, becomes

$$-e\int\bar\psi_p\psi_p\,d^3p = -e\int\bar\psi_x\psi_x\,d^3x$$

So we can say that we have a charge density $j_0 = -e\bar\psi_x\psi_x$, neglecting an infinite constant charge density. We thus have both a localized energy density and a localized charge density, neglecting infinite constants.

We know how the ψ's transform under Lorentz transformations, from the one-electron theory. In the present theory, with second quantization, they transform in the same way. If we now make Lorentz transformations, we find that the charge density is associated with a current density

$$j_r = -e\bar\psi_x\alpha_r\psi_x\,.$$

The charge density and current density can be combined into one formula

$$j_\mu = -e\bar\psi_x\alpha_\mu\psi_x\,, \tag{11.9}$$

where α_0 is understood to be the unit matrix. We have dropped the infinite constant added to j_0, which corresponds to the charge density of all the negative-energy electrons.

§12. *The Electron Field: Relativistic Invariance*

Now, starting from classical equations and using the rules of quantizing according to the Fermi statistics, we obtained the quantum theory of Section 11. The question remains whether this theory is really relativistic. The procedure of quantization involves working with one particular Lorentz frame of reference; it is not a procedure that is manifestly Lorentz-invariant, and it becomes necessary to check the Lorentz invariance of the quantum theory by a direct calculation in the quantum theory.

We can do that by a method analogous to what we did for the scalar field quantized by the method of Bose statistics. We take two field quantities at two general points of space-time and see what their anticommutator is. We evidently have

$$[\psi_a(x), \psi_b(x')]_+ = 0$$

(x means here a point in space-time $x = x_0, x_1, x_2, x_3$, in contrast with the suffix x, which refers to just a point in three-dimensionsl space at a time that is understood). Let us put $[\psi_a(x), \bar{\psi}_b(x')]_+ = K_{ab}(x, x')$. Now, we could evaluate this commutator by expressing the two field functions in terms of their Fourier components. We know the anticommutation relations between the Fourier coefficients, so we have enough material to do the calculation. We can get the result more directly by noting that this quantity K has certain properties which are sufficient to define it and that there is a pretty obvious answer to what K must be to have these properties.

The properties of K are:

(1) $K_{ab}(x, x')$ involves the x's and x''s only through $(x_\mu - x_{\mu'})$. It is obvious that it is only the differences that can come in, because the whole thing is invariant under translations in space.

(2) It satisfies the wave equation

$$[\alpha_\mu i\hbar(\partial/\partial x_\mu) - \alpha_m m]_{ab} K_{bc}(x - x') = 0.$$

The operator $[\alpha_\mu i\hbar(\partial/\partial x_\mu) - \alpha_m m]_{ab}$ acts only on the $\psi_b(x)$, whether it occurs on the left or on the right of $\bar{\psi}_c(x)$, and applied to $\psi_b(x)$ it gives 0, because $\psi(x)$ satisfies the wave equation (11.1), so $K(x - x')$ satisfies the wave equation, too.

(3) For $x_0 = x_0'$, $K_{ab}(x, x') = \delta_3(x - x')\, \delta_{ab}$. This third equation is just the anticommutation relation for the field functions at two points at the same time, which we had previously.

K satisfies those three conditions and conversely those three conditions are sufficient to fix K. We can see that in this way (3) fixes K when $x_0 = x_0'$. Now if you keep x_0' fixed and allow x_0 to vary, the wave equation tells you how K varies. So now we have K determined for all x_0 for the given x_0'. Then condition (1) shows you how K varies when x_0' is allowed to vary. These three conditions thus fix K

completely, and if we can write down a K that satisfies those three conditions we know that it is the correct one.

Well, there is a K which pretty obviously satisfies those three conditions:

$$K_{ab}(x, x') = h^{-3}\int\sum\frac{1}{2}\left(1 + \frac{\alpha_r p_r + \alpha_m m}{p_0}\right)_{ab} e^{-i(x-x')\cdot p/\hbar}\, d^3p. \tag{12.1}$$

Here p_0 means $\pm(p^2 + m^2)^{1/2}$ and the summation refers to the sum over the two values for p_0. Now let us check to see if this K does satisfy the three conditions. Well, (1) is obvious; (2) is also fairly obvious, because if we apply the differential operator to K it will bring into the integrand on the left a factor $(p_0 - \alpha_r p_r - \alpha_m m)$, which, multiplied into $[1 + (\alpha_r p_r + \alpha_m m)/p_0]$, gives zero. Let us see whether (3) is satisfied. Put x_0 equal to x_0'. That means that the term involving p_0 in the exponential disappears, and we get

$$\delta_{ab} h^{-3}\int e^{i(x-x',p)/\hbar}\, d^3p = \delta_{ab}\,\delta(x - x').$$

Therefore all three conditions are fulfilled and this K (12.1) is the value of the anticommutator.

Now we want to show that K is Lorentz-invariant. We can write it

$$K_{ab}(x, x') = J_0 + \alpha_r J_r + \alpha_m J_m, \tag{12.2}$$

where

$$\begin{aligned} J_0 &= h^{-3}\int e^{-i(x-x')\cdot p/\hbar}\, d^3p, \\ J_r &= \tfrac{1}{2}h^{-3}\int\sum p_r e^{-i(x-x')\cdot p/\hbar}\,\frac{d^3p}{p_0}, \\ J_m &= \tfrac{1}{2}h^{-3}\int\sum m e^{-i(x-x')\cdot p/\hbar}\,\frac{d^3p}{p_0}. \end{aligned} \tag{12.3}$$

Let us see how these three integrals transform under Lorentz transformations.

Well, let's take J_m. We may write it

$$J_m = h^{-3}\int m e^{-i(x-x')\cdot p/\hbar}\,\delta(p_0{}^2 - p^2 - m^2)\frac{p_0}{|p_0|}\, d^4p,$$

where p_0 is now an independent variable. To see that both expressions for J_m are equivalent we transform the δ-function,

$$\begin{aligned}\delta(p_0{}^2 - p^2 - m^2)\\ &= \delta\{[p_0 + (p^2 + m^2)^{1/2}][p_0 - (p^2 + m^2)^{1/2}]\}\\ &= |p_0 + (p^2 + m^2)^{1/2}|^{-1}\,\delta[p_0 - (p^2 + m^2)^{1/2}]\\ &\quad + |p_0 - (p^2 + m^2)^{1/2}|^{-1}\,\delta[p_0 + (p^2 + m^2)^{1/2}]\\ &= [2(p^2 + m^2)^{1/2}]^{-1}\,\{\delta[p_0 - (p^2 + m^2)^{1/2}]\\ &\quad + \delta[p_0 + (p^2 + m^2)^{1/2}]\}.\end{aligned}$$

So the δ-function gives rise to two contributions according to whether p_0 is $+(p^2 + m^2)^{1/2}$ or $-(p^2 + m^2)^{1/2}$; and that gives you just the two contributions occurring in (12.3). Well, with J_m in this form it is obvious that it is Lorentz-invariant; all four p's are now treated on the same footing, so J_m equals an invariant; we do not need to work it out.

Now, the other J's can be treated in a corresponding way. We can combine them into a single formula for J_μ:

$$J_\mu = h^{-3}\int p_\mu e^{-i(x'-x)\cdot p/\hbar}\,\delta(p_0{}^2 - p^2 - m^2)\frac{p_0}{|p_0|}\,d^4p.$$

This we see equals a four-vector. Well, that tells us how these coefficients J are affected by a Lorentz transformation.

You might, when you look at expression (12.2) for K, think that it is wrong, because, with $\alpha_0 = 1$, it is $K = \alpha_0 J_0 + \alpha_r J_r + \alpha_m J_m$, and you might be inclined to think that there should be $-\alpha_r J_r$ to give a Lorentz-invariant quantity. That plus sign is quite correct, though. We may write

$$K = \alpha_m \alpha_\mu \alpha_m J^\mu + \alpha_m J_m\,. \tag{12.4}$$

You see that this gives the correct signs, because, putting $\mu = 1, 2, 3$, we have in α_μ a quantity that anticommutes with α_m and gives us the minus to compensate for the minus that occurs when we raise the suffix.

Let us apply a Lorentz transformation to K given by (12.4) and see what happens to it. We can make a Lorentz transformation that changes vectors according to $A_\mu{}^* = a_\mu{}^\nu A_\nu$, where the coefficients satisfy the equation $a_\mu{}^\nu a^{\mu\rho} = g^{\nu\rho}$. The inverse transformation reads $A_\nu = a^\mu{}_\nu A_\mu{}^*$. One easily checks that these relationships are consistent with each other and preserve the squared length of the vector.

Now suppose that under this Lorentz transformation ψ transforms into $\psi^* = L\psi$, where L is some 4×4 matrix. We then have $\bar{\psi}^* = \bar{\psi}\bar{L}$. Now K_{ab} transforms like $\psi_a\bar{\psi}_b$, so

$$K^* = LK\bar{L} = L\alpha_m\alpha_\mu\alpha_m\bar{L}J^\mu + L\alpha_m\bar{L}J_m . \qquad (12.5)$$

From one electron theory we know that the quantities $\bar{\psi}\alpha_\mu\psi$ transform like a four-vector. This in fact is the charge-current density four-vector. That means that $\bar{\psi}^*\alpha_\mu\psi^* = a_\mu{}^\nu\bar{\psi}\alpha_\nu\psi$. The left-hand side here is equal to $\bar{\psi}\bar{L}\alpha_\mu L\psi$. This result holds for any wave function ψ and we get, therefore,

$$\bar{L}\alpha_\mu L = a_\mu{}^\nu\alpha_\nu . \qquad (12.6)$$

Now we also know that $\bar{\psi}\alpha_m\psi$ is an invariant and is equal to

$$\bar{\psi}^*\alpha_m\psi^* = \bar{\psi}\bar{L}\alpha_m L\psi,$$

so

$$\bar{L}\alpha_m L = \alpha_m . \qquad (12.7)$$

Equations (12.6) and (12.7) have $\bar{L}$ on the left and L on the right, and we want to have some equations in which it is the other way around, L on the left and $\bar{L}$ on the right, to evaluate (12.5). Multiply (12.7) by $L\alpha_m$ on the left; we get $L\alpha_m\bar{L}\alpha_m L = L$. We can cancel L on the right on both sides, because it is a matrix without vanishing determinant. Then we can transfer α_m to the other side and get

$$L\alpha_m\bar{L} = \alpha_m . \qquad (12.8)$$

Multiply (12.6) by $L\alpha_m$ on the left and $\alpha_m\bar{L}$ on the right:

$$L\alpha_m\bar{L}\alpha_\mu L\alpha_m\bar{L} = \alpha_\mu{}^\nu L\alpha_m\alpha_\nu\alpha_m\bar{L}.$$

Using (12.8) we get

$$\alpha_m\alpha_\mu\alpha_m = a_\mu{}^\nu L\alpha_m\alpha_\nu\alpha_m\bar{L}. \qquad (12.9)$$

Now we substitute for J^μ in terms of $J^{\mu*}$ in the expression (12.5):

$$K^* = L\alpha_m\alpha_\nu\alpha_m\bar{L}a_\mu{}^\nu J^{\mu*} + L\alpha_m\bar{L}J_m{}^*.$$

Using equations (12.9) and (12.8) we get

$$K^* = \alpha_m\alpha_\mu\alpha_m J^{\mu*} + \alpha_m J_m{}^*.$$

There you see we have K^* of the same form as K, and that proves the Lorentz invariance of the anticommutation relations. The Lorentz invariance of the whole theory is thus established.

§13. *Fields with Interaction: Relativistic Invariance*

As a result of this work we have a satisfactory quantum theory for a scalar field, a field of bosons, or the electron field with Fermi quantization. But we have a satisfactory quantum theory for these fields only by themselves. To get something interesting, we have to put in an interaction.

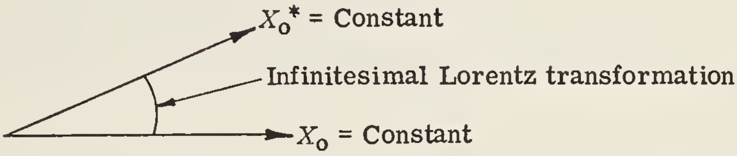
The method of putting in an interaction is to add up the various Hamiltonians for the various fields alone and then add a further term to the Hamiltonian to correspond to the interaction energy. The question is: What shall we take for the interaction energy? Well, we can work from the classical theory, which will suggest an interaction energy for us to take. We can take over this interaction energy from the classical theory, and then we have a quantum theory. We then have to check, from calculations performed entirely in quantum theory, whether it is all right. In particular, we have to check whether our theory is really a Lorentz-invariant theory with the interaction. To do that we must work out the field equations, given by the Heisenberg equations of motion, and check whether they are Lorentz-invariant. Then we have to check that our commutation or anticommutation relations are Lorentz-invariant.

I want now to discuss the question of how we deal with the Lorentz invariance of the commutation or the anticommutation relations when there is interaction. The work in Sections 10 and 12 shows that it is all right when there is no interaction; we have to generalize this work to take interaction into account.

Well, you might say, let us use the same method when there is interaction. That is going to be extremely difficult, because we have in the commutator or anticommutator two different times; and if we want to compare quantities referring to two different times we have to integrate the Heisenberg equations of motion. To integrate the Heisenberg equations of motion when there is interaction is a very complicated process. We need some other method.

Instead of taking two arbitrary points in space-time, which is really more than necessary, let us just start from the commutation relations connected with field quantities at one particular time, that is, field quantities on a three-dimensional surface x_0 = constant in space-time, and let us apply an infinitesimal Lorentz transformation, so as to go over to another three-dimensional surface inclined with respect to the first one. Let us then see what are the commutation or anticommutation relations for the field quantities on the new three-dimensional surface. If we can check that they are all right, we should know that our commutation or anticommutation relations remain invariant under infinitesimal Lorentz transformations; and if they are invariant under infinitesimal Lorentz transformations, they will be invariant under finite ones.

We start off with our various field quantities on a surface x_0 = constant in space-time and we make an infinitesimal Lorentz transformation to the surface $x_0{}^*$ = constant as in this space-time diagram:



We have known commutation or anticommutation relations between the quantities for a given x_0, which are assumed not to be affected by the interaction. We shall calculate from them the commutation or anticommutation relations at $x_0{}^*$ = constant.

The Lorentz transformation will involve a change in the time coordinate $x_0{}^* = x_0 + \epsilon V_r x_r$, where ϵ is infinitesimal and ϵV_r gives the relative velocity of the two frames. The V_r's are c-numbers. We shall neglect ϵ^2. Now we take two field quantities K and λ; they may be U or V or ψ_a or $\bar{\psi}_a$. We have

$$K_x(x_0{}^*) = K_x(x_0) + (x_0{}^* - x_0)\frac{dK}{dx_0} = K_x(x_0) + \epsilon V_r x_r [K_x, H],$$

where H is the total Hamiltonian including interaction. With λ we do the same thing:

$$\lambda_{x'}(x_0{}^*) = \lambda_{x'}(x_0) + \epsilon V_s x_s' [\lambda_{x'}, H].$$

Now let us work out the Poisson bracket for those two quantities.

$$\begin{aligned}[K_x(x_0{}^*), \lambda_{x'}(x_0{}^*)] &= [K_x(x_0), \lambda_{x'}(x_0)] + \epsilon V_r x_r [[K_x, H], \lambda_{x'}] \\ &\quad + \epsilon V_s x_s' [K_x, [\lambda_{x'}, H]] \\ &= [K_x(x_0), \lambda_{x'}(x_0)] + \epsilon V_r (x_r' - x_r)[K_x, [\lambda_{x'}, H]] \\ &\quad + \epsilon V_r x_r [[K_x, \lambda_{x'}], H], \qquad (13.1)\end{aligned}$$

where the last line was obtained with the help of Jacobi's identity.

Now there are some occasions when we are interested not in the Poisson bracket of two field quantities but in their anticommutator; if they are both ψ's or one of them is ψ and the other $\bar{\psi}$. So we have to do the calculation also for the anticommutator. We have

$$[K_x(x_0{}^*), \lambda_{x'}(x_0)^*]_+ = [K_x(x_0), \lambda_{x'}(x_0)]_+ \\ + \epsilon V_r x_r [[K_x, H], \lambda_{x'}]_+ + \epsilon V_r x_r' [K_x, [\lambda_{x'}, H]]_+ .$$

In this formula you note we have sometimes the anticommutator and sometimes the Poisson bracket; where we have the Hamiltonian coming in, we have to keep to the Poisson bracket. The formula involves a mixture of anticommutators and Poisson brackets, but we can handle it as before.

$$\begin{aligned}[K_x(x_0{}^*), \lambda_{x'}(x_0{}^*)]_+ = {} & [K_x(x_0), \lambda_{x'}(x_0)]_+ \\ & + \epsilon V_r(x_r' - x_r)[K_x, [\lambda_{x'}, H]]_+ \\ & + \epsilon V_r x_r \{[K_x, [\lambda_{x'}, H]]_+ + [[K_x, H]\lambda_{x'}]_+\} . \end{aligned} \tag{13.2}$$

Using the notation of the footnote on page 37, we have the formula

$$[[a, b]_+, c]_- = [a, [b, c]_-]_+ + [[a, c]_-, b]_+ . \tag{13.3}$$

You can check it just by multiplying it out with straightforward non-commutative algebra; you find that everything fits. We have here a formula for anticommutators analogous to the Jacobi identity. With the help of it we can write

$$\{[K_x, [\lambda_{x'}, H]]_+ + [[K_x, H], \lambda_{x'}]_+\} = [[K_x, \lambda_{x'}]_+ H].$$

The expression (13.2) then becomes similar to (13.1). It becomes possible to combine them into a single formula:

$$\begin{aligned}[K_x(x_0{}^*), \lambda_{x'}(x_0{}^*)]_\pm = {} & [Kx(x_0), \lambda x'(x_0)]_\pm \\ & + \epsilon V_r(x_r' - x_r)[K_x, [\lambda_{x'}, H]]_\pm, \\ & + \epsilon V_r x_r [[K_x, \lambda_{x'}]_\pm, H]. \end{aligned} \tag{13.4}$$

This one formula sums up the two results for the Poisson bracket or commutator and for the anticommutator.

K and λ so far have been quite arbitrary field quantities. I am now going to take them to be basic field variables. Take K_x to be one of V_x, U_x, ψ_{ax}, $\bar{\psi}_{ax}$ and $\lambda_{x'}$ to be one of $V_{x'}$, $U_{x'}$, $\psi_{ax'}$, $\bar{\psi}_{ax'}$. Under those conditions $[K_x, \lambda_{x'}]_\pm = 0$ or $\pm\delta_3(x - x')$. In fact, all the basic field quantities we have been concerned with satisfy commutation or anticommutation relations of this type. We either have naught on the right side or we have the δ-function or minus the δ-function. Thus $[K_x, \lambda_{x'}]_\pm = c$-number and $[[K_x, \lambda_{x'}]_\pm, H] = 0$. If we took more complicated local things, e.g., if we took K and λ to be functions of U and V, that wouldn't hold, of course. But we are only interested in the commutation relations of the basic field quantities.

The last term in (13.4) has thus been cut out. Now let's see what we are left with:

$$\begin{aligned}[K_x(x_0^*), \lambda_{x'}(x_0^*)]_\pm = {} & [K_x(x_0), \lambda_{x'}(x_0)]_\pm \\ & + \epsilon V_r(x_r' - x_r)[K_x, [\lambda_{x'}, H_P]]_\pm \\ & + \epsilon V_r(x_r' - x_r)[K_x, [\lambda_{x'}, H_Q]]_\pm,\end{aligned}$$

where I have taken into account that H is equal to the proper energy H_P plus an interaction energy H_Q. Proper energy means the energy of the fields by themselves, without any interaction; and H_Q is the interaction energy which is added to H_P.

We now see the condition for the commutation or anticommutation relations to remain Lorentz-invariant. We know that they are Lorentz-invariant when there is no interaction; that we proved previously. So we want the interaction not to disturb the commutation or anticommutation relations, and that means that

$$\begin{aligned}(x_r' - x_r)[K_x, [\lambda_{x'}, H_Q]]_\pm &= 0, \\ [K_x, [\lambda_{x'}, H_Q]]_\pm &= \text{multiple of } \delta_3(x - x').\end{aligned} \tag{13.5}$$

Take $H_Q = \int W_{Qx}\, d^3x$, where W_{Qx} is the interaction energy density. I am assuming a local theory in which the interaction is given by a localized energy density, integrated over three-dimensional space. We have now to choose this function W_Q so as to satisfy the condition (13.5). We see that the condition is satisfied if W_{Qx} equals a function of U_x, V_x, ψ_{ax}, $\bar{\psi}_{ax}$ with no spatial derivatives. In general, of course, you might have an interaction energy density involving spatial derivatives, derivatives with respect to x_1, x_2, x_3; but suppose we take the case when this is not so. In that case you see that the Poisson bracket of H_Q with $\lambda_{x'}$ will involve the field quantities at x' without any derivatives, and the commutator or anticommutator of this Poisson bracket with K_x will give some multiple of the δ-function without derivatives of the δ-function.

So here is quite a substantial class of interaction-energy densities we can take, which we can be confident will not destroy the Lorentz-invariance of the commutation or anticommutation relations. This class is sufficient for practical purposes. Theoretically one can generalize a little bit but that generalization hasn't turned out to be useful.

That is a formal proof of Lorentz invariance. It could be criticized on the grounds that we are not making proper use of the δ-function in this kind of work, because what we are doing is taking two field points close together and expanding the commutator or anticommutator in a Taylor series in $(x - x')$. If these two field points are very close together, $[K_x, \lambda_{x'}]_\pm$ cannot in general be expanded as a Taylor series in $(x - x')$. It is obvious that it cannot, because something very singular happens when one of the two points x and x' is on the light cone of the other.

The above use of the δ-function is not a correct one, but we can put the calculation in order in a rather simple way. We form $\int a_x K_x \, d^3x$ and $\int b_{x'} \lambda_{x'} \, d^3x_{'}$, where a and b are continuous c-number functions of the three x's, and then we form $[\int a_x K_x \, d^3x, \int b_{x'} \lambda_{x'} \, d^3x']$. That is something that does not involve any δ-functions. It just involves integration over a and b, and it is quite permissible to calculate how that quantity transforms when we make a Lorentz transformation, using a Taylor expansion to connect the things on $x_0{}^*$ with the things on x_0. We can then go through work that would be exactly the same as the work above with all our equations multiplied by a_x, $b_{x'}$ and integrated over x or x'. Then, of course, there wouldn't be any δ-functions coming in at all.

You can put the calculation right in that way. It would be rather tedious to write out all these equations with these extra factors coming in—the equations are quite long as it is; but you see that it will be quite possible, and then we should not be making any unjustifiable Taylor expansions. So although the calculation as I gave it is not really justifiable as it stands, it can be put in order in a fairly easy way. One gets the same conclusion that, provided the interaction energy satisfies the above-mentioned condition, the commutation or anticommutation relations are Lorentz-invariant.

§14. *Constraints*

I now want to go on to another subject, the question of constraints in the quantum theory. When we work from a classical theory and put it in the Hamiltonian form, we might very well get constraints appearing—equations involving the q's and p's only. These constraints are of two kinds: first class and second class.

The second-class constraints can be eliminated. With the help of a suitable modification in the definition of Poisson brackets, the second-class constraint equations can be taken to be strong equations or definitions. We are left with first-class constraints. We have now to consider the general problem of a Hamiltonian with first-class constraints. How are we to take that over into the quantum theory?

A full account of classical constraints is given in my book, referred to on page 42. The quantum theory has to be reformulated in terms of the Heisenberg picture, without the help of the Schrödinger picture.

The constraint equations are weak equations, written $\phi_m \approx 0$, which means they must be used in accordance with certain rules. I'll first give the classical rules.

Constraints can be added. They can be multiplied by factors: the factors can be functions of the q's and p's. We must not use them in Poisson brackets, i.e., $[\phi_m, f] \not\approx 0$ in general.

Now we have to set up suitable quantum rules. We maintain the addition rule. Now what about multiplying? If we can multiply them by factors on both sides, we can form a quantum Poisson bracket; and that we must not do, as we are not allowed to form Poisson brackets in the classical theory. How can we avoid the possibility of forming Poisson brackets? We do that by adopting the rule that we can multiply the weak equations by factors only on one side, for example, the left side. So we take as our second rule that we can multiply by factors, functions of the q's and p's, on the left and that we cannot multiply by factors on the right in general. So we cannot form Poisson brackets.

These are the rules we take to be the rules for working with weak equations or constraints in the quantum theory. These rules destroy the symmetry between left and right multiplication. This symmetry between left and right, which we have had all the time in our quantum theory up to the present, gets destroyed when we have a quantum theory with constraints in it.

When we have weak equations in the classical theory we need certain consistency conditions between them. We have the classical consistency conditions

$$[\phi_m, \phi_n] \approx 0. \tag{14.1}$$

These consistency conditions we assume to be the same in the quantum theory.

In the classical theory the consistency conditions may be written as strong equations,

$$[\phi_m, \phi_n] = a_{mnj}\phi_j. \tag{14.2}$$

Anything that is weakly equal to zero has to be strongly equal to some linear combination of the ϕ's, because the ϕ's are the only independent quantities weakly equal to zero. In the quantum theory the consistency conditions may also be written as strong equations like (14.2), but now the coefficients have to be on the left. In the classical theory it doesn't matter whether these coefficients a_{mnj} are on the left or right, but when we go over to the quantum theory we must insist that the coefficients are on the left.

In the classical theory we have the result that the Hamiltonian is first class. This means that

$$[\phi_m, H] \approx 0. \tag{14.3}$$

In the quantum theory we maintain this condition. These weak equations can be written as strong equations:

$$[\phi_m, H] = b_{mn}\phi_n ; \tag{14.4}$$

and in the quantum theory we must again insist that the coefficients shall be on the left.

Those are the rules by which one handles constraints in the quantum theory. Now, you may be rather perturbed by our giving up the symmetry between left and right multiplication; you will say, does it matter that we have to give up this symmetry? It matters in one respect, with regard to our definition of real q-numbers. We shall have to go into this question of reality again.

Before we do that, I should like to develop the theory further. Let us use the constraints $\phi_m \approx 0$ to impose conditions on the ket vector $|P\rangle$:

$$\phi_m|P\rangle = 0. \tag{14.5}$$

The question arises: Can we find a ket to satisfy all those conditions? Let us work on those conditions a little and see whether we get an inconsistency. From $\phi_m|P\rangle = 0$ and $\phi_n|P\rangle = 0$ we can deduce $\phi_n\phi_m|P\rangle = 0$ and $\phi_m\phi_n|P\rangle = 0$; therefore $[\phi_n, \phi_m]|P\rangle = 0$. This condition is not independent of the previous ones. From (14.2) it is just $a_{nmj}\phi_j|P\rangle = 0$. So working with the restrictions (14.5) on the ket in this way we cannot get new restrictions, and under those conditions one can assume that the ket $|P\rangle$ exists.

With our rules for working with the constraints we can thus use the constraints to define kets by (14.5), but we cannot use them to define bras; it just wouldn't work out, because the factors would be on the wrong side. A ket that satisfies these conditions is called a *physical ket*. That's a definition.

We now define a q-number to be physical if, when it is multiplied into any physical ket, it gives another physical ket. The physical kets form a subspace in all the ket vector space and the physical q-numbers are operators that operate in this subspace. If α is a physical q-number we have for any ϕ_n, $\phi_n\alpha|P\rangle = 0$, and hence $(\phi_n\alpha - \alpha\phi_n)|P\rangle = 0$ for all physical kets $|P\rangle$; so

$$[\phi_n, \alpha] \approx 0. \tag{14.6}$$

Conversely, the conditions (14.6) ensure that α is a physical q-number.

Perhaps I should say at this point, to give you some physical insight into all this, that we shall later be applying this theory to electrodynamics and we shall then find that the quality of being physical is just the quality of being gauge-invariant. The physical q-numbers correspond to quantities that are gauge-invariant and the physical kets are kets that are gauge-invariant.

You see that if α and β are physical so is $\alpha + \beta$, and of course $\alpha\beta$ and $\beta\alpha$ are also physical. The Poisson bracket of α, β is also physical. So the physical operators form a ring. You can add and multiply them freely and get other physical operators from them.

Equations (14.3) shows that the Hamiltonian H is physical. We now see that, if α is physical, so is $\dot{\alpha} = [\alpha, H]$. Thus if α is initially physical, it always remains physical.

Now let us consider the question of conjugate complex q-numbers. At this stage we see that we can define the conjugate complex of a q-number if it is a physical q-number. We can define it in the way we did in Section 3 for the case of no constraints. We assume that every physical α has a complex conjugate $\bar{\alpha}$ that is also physical, and we assume that

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta} \qquad \text{and} \qquad \overline{\alpha\beta} = \bar{\beta}\,\bar{\alpha}. \tag{14.7}$$

Each ϕ_m itself is physical, according to the definition. It is to be counted as real, because it has the value zero for all physical kets.

These assumptions for conjugate complex q-numbers cannot be extended in general to nonphysical q-numbers. One can see this as follows. We saw that ϕ_m is real. The equation expressing this should be written as a weak equation,

$$\phi_m \approx \bar{\phi}_m .$$

Similarly, for any q-number α, $\alpha\phi_m$ is weakly real,

$$\alpha\phi_m \approx \overline{\alpha\phi_m} .$$

Take α to be real and apply the second of the rules (14.7). We get

$$\alpha\phi_m \approx \phi_m\alpha.$$

For a general q-number α this conclusion would be false, but for a physical q-number α it is all right, according to (14.6). So we have a consistent scheme only as long as we restrict our ideas of conjugate complex q-numbers to the physical q-numbers. That is really sufficient for applications of the theory.

The importance of a definition for reality is that it will enable us to deduce that real physical q-numbers have real eigenvalues, and that is most important when we connect our theory with measurements. The fact that we don't have a definition of reality for the nonphysical dynamical variables doesn't matter, because they are not quantities which can be measured. Well, this is the general situation with regard to constraints in the quantum theory and also with regard to conjugate complex q-numbers.

There is a special case where we can introduce the notion of the conjugate complex for nonphysical q-numbers. That is the special case which occurs when the coefficients a_{mnj}, b_{mn}, in (14.2) and (14.4) commute with all the ϕ's, for example, if they are c-numbers. Then

$$[\phi_m , \phi_n] = a_{mnj}\phi_j = \phi_j a_{mnj} ,$$
$$[\phi_m , H] = b_{mn}\phi_n = \phi_n b_{mn} .$$

If we have this special case, then in our basic equations there is no dissymmetry between left and right multiplication and we could continue to develop the theory symmetrically between left and right multiplication. We can then maintain the definition for conjugate complex q-numbers even when the q-numbers are not physical. So we have this special case in which it is possible to preserve the notions of conjugate complex and reality for nonphysical q-numbers.

It so happens that this special case is realized in the case of electrodynamics. Electrodynamics is one of the important applications of constraints. Another important application is to the gravitational field or to any theory involving curvilinear coordinates. We have then constraints to express the conditions that our equations must remain invariant under transformations of coordinates. In case of these constraints associated with curvilinear coordinates, we do not have the special case, and then we do not have the possibility of defining real q-numbers among the nonphysical q-numbers. We can define real q-numbers then only for the physical ones, which would be the ones independent of the system of coordinates.

§15. *The Electromagnetic Field without Charges*

I should like now to proceed to develop electrodynamics. Let us first take the electromagnetic field by itself. We can use the classical theory as a basis for getting a quantum theory. There are different forms in which the classical theory can be put. I propose to work now with Fermi's form, in which we use the constraint

$$\frac{\partial A_\mu}{\partial x_\mu} \approx 0. \tag{15.1}$$

This is the form that is mostly used in practice; it has the advantage that there's rather more symmetry between the four coordinates of space-time than with other forms.

The classical action density $\mathfrak{L}$ for this form of electrodynamics is given by

$$4\pi\mathfrak{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}A_{\mu}{}^{,\mu}A_{\nu}{}^{,\mu}, \tag{15.2}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Well let us work out what we get with it. First, we see that it can be transformed into

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}A_{\mu}A_{\nu}{}^{,\nu\mu} + \text{perf. diff.}$$

Now we can make a further change to

$$-\tfrac{1}{2}A_{\mu,\nu}(A^{\mu,\nu} - A^{\nu,\mu}) - \tfrac{1}{2}A_{\mu}{}^{,\nu}A_{\nu}{}^{,\mu}.$$

plus another perfect differential, and we are left with

$$4\pi\mathfrak{L} = -\tfrac{1}{2}A_{\mu,\nu}A^{\mu,\nu} + \text{perf. diff.} \tag{15.3}$$

The perfect differential does not affect the field equations and can be neglected.

Our action density is now such that for each value of μ it is just the action density (9.1) for a scalar field with no mass term, except that there is a different sign for $\mu = 0$. We have here the action density for four scalar fields. So far as the action density is concerned, and ignoring the constraints (15.1), we have four independent scalar fields; they just happen to be labeled $\mu = 0, 1, 2, 3$, but that labeling of the four fields as the four components of a four-vector is not relevant to the derivation of equations of motion from the action principle. You see at once that the equations of motion are just $\Box A_\mu = 0$, like (9.2). It is obvious that these equations of motion are consistent with the supplementary condition (15.1). So what we do is work with the action density (15.2) or (15.3) and bring in the Lorentz equation (15.1) as an extra condition which we impose on our dynamical variables; and we see that this extra condition is consistent with the equations of motion.

Let us now proceed to apply the usual method for going over to the Hamiltonian. We introduce the momentum variables $-4\pi B^\mu$ conjugate to A_μ, as with the scalar field in Section 9. We put the minus sign here so as to have $4\pi B_\mu$ conjugate to A_μ for the case of $\mu = 1, 2, 3$. The B_μ then have the values $B_\mu = A_\mu{}^{,0}$ and satisfy the Poisson-bracket relations

$$[A_{\mu x}, B_{\nu x'}] = -4\pi g_{\mu\nu}\delta_3(x - x'). \tag{15.4}$$

We now get for the Hamiltonian (call it H_F),

$$H_F = -(8\pi)^{-1}\int (B_\mu B^\mu + A_\mu{}^{,r}A^{\mu,r})\, d^3x, \tag{15.5}$$

which is just the Hamiltonian (9.3) for four scalar fields with a minus sign for the one with $\mu = 0$. The degree of freedom associated with A_0, B_0 give a negative contribution to the energy. If we take $\mu = 1, 2,$ or 3, we get the usual positive contribution.

So far this is just a treatment of four scalar fields. We now have to bring in the restriction on the potentials (15.1). Let us see what conditions they give us on the Hamiltonian variables at a certain time. These conditions will be the Hamiltonian constraints. At one particular time we must take

$$A_{\mu}{}^{,\mu} = 0 \tag{15.6}$$

and

$$\frac{\partial}{\partial x_0}(A_{\mu}{}^{,\mu}) = 0. \tag{15.7}$$

These will be independent if we think of them just as conditions holding at one particular time. So there we have two independent conditions which we are imposing on the A's. They are all the independent ones, because the further condition $(\partial^2/\partial x_0 \partial x_0) A_{\mu}{}^{,\mu} = 0$ is a consequence of the previous ones and the field equations $\Box A_{\mu} = 0$. This further condition and all the further ones we get by successive differentiations with respect to time are not independent but are consequences of the equations of motion.

The condition (15.6), expressed in terms of Hamiltonian variables, gives

$$B_0 + A_r{}^{,r} \approx 0. \tag{15.8}$$

The condition (15.7) may be written with the help of the field equation $\Box A_0 = 0$ as $A_0{}^{,rr} + A_r{}^{,0r} = 0$. This is just the Maxwell equation $\operatorname{div} \mathcal{E} = 0$. Expressed in terms of the Hamiltonian variables it reads

$$B_r{}^{,r} + A_0{}^{,rr} \approx 0. \tag{15.9}$$

We have in (15.8) and (15.9) all the Hamiltonian constraints. Each of these equations represents a threefold infinity of constraints, one for each point x_1, x_2, x_3 in three-dimensional space. We may write them

$$\phi_1 \approx 0, \qquad \phi_2 \approx 0,$$

where ϕ_1 and ϕ_2 are defined by

$$\phi_1 = B_0 + A_r{}^{,r}, \qquad \phi_2 = B_r{}^{,r} + A_0{}^{,rr}. \tag{15.10}$$

To check the consistency conditions (14.1), we have to check that

$$[\phi_{1x}, \phi_{1x'}] \approx 0, \qquad [\phi_{1x}, \phi_{2x'}] \approx 0, \qquad [\phi_{2x}, \phi_{2x'}] \approx 0. \tag{15.11}$$

These come out very directly. In fact, they hold as strong equations, with all the coefficients a_{mnj} of (14.2) vanishing.

We quantize that theory by treating the four A's like four scalar fields and using the Bose statistics. The four fields are quite detached as long as one does not use the constraints. The constraints are the only equations which give any relationship between them. If we forget about the constraints for the time being, we can quantize the theory just like four scalar fields.

We introduce Fourier coefficients $A_{\mu k}$ just as we did before in the case of the scalar field. We take the classical expression (15.5) for the Hamiltonian into the quantum theory, and then if we put it in terms of the Fourier coefficients we get

$$H_F = -4\pi^2 \int k^2 A_{\mu k} \overline{A}_k{}^{\mu} \, d^3{}_k + \text{infinite } c\text{-number},$$

corresponding to (9.16). The Fourier transformation gives directly the symmetrical product $\frac{1}{2}(A\overline{A} + \overline{A}A)$, and to replace it by the unsymmetrical one with the $\overline{A}$'s all on the right involves the introduction of an infinite c-number.

The point to be noted about this Hamiltonian is that the contribution for the zero degree of freedom is negative. I'd like to write that out explicitly:

$$H_F = 4\pi^2 \int k^2 (A_{1k}\overline{A}_{1k} + A_{2k}\overline{A}_{2k} + A_{3k}\overline{A}_{3k} - A_{0k}\overline{A}_{0k}) \, d^3k$$
$$+ \text{ infinite constant.} \qquad (15.12)$$

We have this rather peculiar thing: a negative energy associated with the A_0 waves.

The commutation relations between the Fourier coefficients are, corresponding to (9.12) and (9.13),

$$[A_{\mu k}, A_{\nu k'}] = 0, \qquad [A_{\mu k}, \overline{A}_{\nu k'}] = \frac{-ig_{\mu\nu}}{4\pi^2 |k|} \delta_3(k - k'). \qquad (15.13)$$

They lead to the commutation relation between the A's at two general points in space-time:

$$[A_\mu(x), A_\nu(x')] = g_{\mu\nu} \Delta(x - x'), \qquad (15.14)$$

corresponding to (10.6).

We get the above results simply from quantizing the four scalar fields, and now we have to bring in the constraint equations. We must first check that they are consistent with each other. We already did that checking in our classical work, and it goes the same in the quantum theory. Alternatively, we may check the consistency directly from (15.14). We can do that by forming

$$\begin{aligned}\left[\frac{\partial A_\mu(x)}{\partial x_\mu}, \frac{\partial A_\nu(x')}{\partial x_\nu{}'}\right] &= \frac{\partial^2}{\partial x_\mu\,\partial x_\nu{}'}[A_\mu(x), A_\nu(x')] \\ &= \frac{\partial^2}{\partial x_\mu\,\partial x_\nu{}'} g_{\mu\nu}\,\Delta(x - x') \\ &= -\frac{\partial^2}{\partial x_\mu\,\partial x_\nu} g_{\mu\nu}\,\Delta(x - x') \\ &= 0,\end{aligned}$$

from (10.7).

The various Poisson brackets (15.10) all vanish strongly. That's really more than is necessary. We need them to vanish only weakly and find them to vanish strongly, with all the coefficients a_{mnj} zero. So we have the special case where it is possible to preserve the ideas of conjugate complex q-numbers also for the nonphysical ones.

Well let us find out now which are the physical dynamical variables. A physical dynamical variable has to commute with all these $\partial A_\mu/\partial x_\mu$ quantities; at any rate it must commute weakly with them. Suppose we form

$$\left[\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}, \frac{\partial A_\sigma(x')}{\partial x_\sigma{}'}\right].$$

Well we can work that out as

$$\begin{aligned}&\frac{\partial^2}{\partial x_\mu\,\partial x_\sigma{}'}[A^\nu(x), A_\sigma(x')] - \frac{\partial^2}{\partial x_\nu\,\partial x_\sigma{}'}[A^\mu(x), A_\sigma(x')] \\ &\qquad = \frac{\partial^2}{\partial x_\mu\,\partial x_\sigma{}'} g_\sigma{}^\nu\,\Delta(x - x') - \frac{\partial^2}{\partial x_\nu\,\partial x_\sigma{}'} g_\sigma{}^\mu\,\Delta(x - x') \\ &\qquad = 0,\end{aligned}$$

on account of the antisymmetry of $\Delta(x - x')$ between x and x'.

Now $(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)$ give the electromagnetic fields $\mathcal{E}$ and $\mathcal{H}$. We see that the electric and magnetic field quantities are physical in the sense in which I have defined physical quantities. The potentials are not physical. That corresponds to the physical quantities being gauge-invariant quantities.

To express the constraints in terms of the Fourier coefficients, we have

$$A_\mu(x) = \int (A^c_{\mu k} e^{ik\cdot x} + \overline{A}^c_{\mu k} e^{-ik\cdot x})\, d^3k.$$

We now form

$$\frac{\partial A_\mu(x)}{\partial x_\mu} = -i \int k^\mu (A^c_{\mu k} e^{ik\cdot x} - \overline{A}^c_{\mu k} e^{-ik\cdot x})\, d^3x.$$

The constraints (15.1) give $k^\mu A^c_{\mu k} \approx 0$ and $k^\mu \overline{A}^c_{\mu k} \approx 0$. We can pass from the constant Fourier coefficients to the time-dependent ones because the same time factor is involved for all four of them. So we may write the constraint equations between the Fourier components in the form

$$k^\mu A_{\mu k} \approx 0, \qquad k^\mu \overline{A}_{\mu k} \approx 0. \tag{15.15}$$

One can separate the waves A_1, A_2, A_3 into longitudinal and transverse parts. If we consider waves moving in direction 3, then A_{3k} is a longitudinal Fourier component and A_{1k}, A_{2k} are the transverse Fourier components. It is convenient to count A_0 also as a longitudinal wave. That separation is not relativistic. If we make a Lorentz transformation the separation would not be the same, but in spite of its not being Lorentz-invariant it is a useful separation for the discussion of various features of the theory.

If we take waves moving in the direction of the axis 3, so that $k_1 = k_2 = 0, k_3 = k_0$, the equations (15.15) give

$$A_{0k} - A_{3k} \approx 0, \qquad \overline{A}_{0k} - \overline{A}_{3k} \approx 0.$$

We shall make some deductions from these weak equations, remembering that we can multiply them only by factors on the left. We have

$$(\overline{A}_0 + \overline{A}_3)(A_0 - A_3) + (A_0 + A_3)(\overline{A}_0 - \overline{A}_3) \approx 0.$$

This gives us $\overline{A}_0 A_0 + A_0 \overline{A}_0 - \overline{A}_3 A_3 - A_3 \overline{A}_3 \approx 0$. We have $\overline{A}_0 A_0 - A_0 \overline{A}_0$ equal to some c-number; this happens to be infinite if we take exactly the same k values in both factors, but that doesn't bother us, because in any case it cancels with $\overline{A}_3 A_3 - A_3 \overline{A}_3$. The result is that $2(\overline{A}_0 A_0 - A_3 \overline{A}_3) \approx 0$.

We may now write (15.12)

$$H_F = 4\pi^2 \int k^2 (A_{1k}\bar{A}_{1k} + A_{2k}\bar{A}_{2k} + A_{3k}\bar{A}_{3k} - \bar{A}_{0k}A_{0k})\, d^3k, \tag{15.16}$$

ignoring an infinite constant. This H_F has no longitudinal energy when applied to a physical ket. I have proved it only for waves moving in the direction of axis 3. But the same thing is true for waves moving in any direction, because with this way of writing H_F there's nothing to distinguish axis 3 from the general axis in three-dimensional space. So we adopt this form for H_F, and then for any physical ket the longitudinal part of the energy vanishes.

This mysterious negative energy now gets accounted for. The physical kets are the only ones we shall want to make use of in the interpretation of the theory; and for these kets, the negative energy associated with the A_0 waves is just compensated for by the longitudinal part of the energy associated with the A_1, A_2, A_3 waves.

We can express $A_{rk} = A^T_{rk} + A^L_{rk}$, the transverse and longitudinal parts, respectively. There are just two independent components among the transverse part, because $k^r A^T_{rk} = 0$. We have, with H_F given by (15.16),

$$H_F \approx 4\pi^2 \int k^2 A^T_{rk}\bar{A}^T_{rk}\, d^3k.$$

This H_F applied to $|0\rangle$ gives 0, where $|0\rangle$ is defined to satisfy $\bar{A}^T_{rk}|0\rangle = 0$ and also to be a physical ket, which means

$$k^\mu A_{\mu k}|0\rangle = 0, \qquad k^\mu \bar{A}_{\mu k}|0\rangle = 0.$$

§16. *Quantum Electrodynamics*

Let us consider how the theory of the electromagnetic field gets altered when there are charges present. Beginning with the classical theory, we have to work with an action I equal to the action of the field plus the action of the matter,

$$I = \int (\mathcal{L}_F + \mathcal{L}_M)\, d^4x,$$

where $\mathcal{L}_F$ is the action density for the field alone, given by (15.2) or

(15.3), and $\mathcal{L}_M$ is the action density for the matter, including the action density for the interaction between the matter and the electromagnetic field.

In all the theories of practical interest $\mathcal{L}_m$ involves the electromagnetic field variables only through involving the potentials and not through involving any derivatives of the potentials. I shall assume that now. There's no reason why one should not consider a more general case in which derivatives of the potentials appear, but this more general case has not yet turned out to be of any practical importance.

Let us continue with our general methods and leave it undetermined how the matter variables occur in $\mathcal{L}_m$ or just what the matter variables are. We shall require, of course, only that $\mathcal{L}_m$ shall be invariant under Lorentz transformations, and then we have a relativistic theory.

If we vary A^μ we get, with the form (15.3) for $\mathcal{L}_F$,

$$\begin{aligned}\delta I &= \int\left(-(4\pi)^{-1}A_{\mu,\nu}\,\delta A^{\mu,\nu} + \frac{\partial\mathcal{L}_m}{\partial A^\mu}\,\delta A^\mu\right)d^4x\\ &= \int\left((4\pi)^{-1}A_{\mu,\nu}{}^{,\nu} + \frac{\partial\mathcal{L}_m}{\partial A^\mu}\right)\delta A^\mu\,d^4x;\end{aligned}$$

so $\delta I = 0$ gives

$$A_{\mu,\nu}{}^{,\nu} - 4\pi j_\mu = 0,$$

where $j_\mu = -\partial\mathcal{L}_m/\partial A^\mu$. With the help of the Lorentz condition (15.1), it may be written

$$F_{\mu\nu}{}^{,\nu} + 4\pi j_\mu = 0. \tag{16.1}$$

We get Maxwell's equations if we interpret this j_μ as a current density, and we see how the current density is determined from the action principle: It is just the derivative of $\mathcal{L}_m$ with respect to A^μ, with a minus sign.

We can proceed to pass to the Hamiltonian formulation by the usual method. With $\mathcal{L}_M$ not involving derivatives of the A_μ, the introduction of momenta conjugate to A_μ goes the same as in the case of no charges. We have again $B_\mu = A_\mu{}^{,0}$ satisfying the Poisson-bracket relations (15.4). Denoting by q_n the dynamical coordinates needed to describe the matter, whatever they may be, and by p_n their conjugate momenta, we get for the Hamiltonian

$$H = \int\left(-\frac{B_\mu}{4\pi}A^{\mu,0} + p_n q_n{}^{,0} - \mathcal{L}_F - \mathcal{L}_m\right)d^3x.$$

We may conveniently write it as a sum of two parts: $H = H_F + H_M$. H_F comes from the first and third terms and is given by just the same equation as in the case of no charges, (15.5). H_m consists of the remaining terms

$$H_m = \int (p_n \dot{q}_n - \mathfrak{L}_m)\, d^3x.$$

That's as far as one can go in completely general terms. One may specialize by assuming that $\mathfrak{L}_m$ is linear in the potentials; that is, those terms that do contain potentials are linear in them. In the important examples it is linear in the potentials. Then j_μ is independent of the A^μ, and we have the current density involving only the matter variables. Under these conditions, of course,

$$\mathfrak{L}_m = -A_\mu j^\mu + \text{terms independent of the } A_\mu .$$

If j_μ does not contain velocities, H_M consists of an interaction part,

$$H_Q = \int A_\mu j^\mu \, d^3x, \tag{16.2}$$

plus a part involving only the matter variables.

In quantum electrodynamics we consider the electromagnetic field interaction with electrons and positions. We have already considered the electromagnetic field by itself in Section 15 and electrons and positrons by themselves in Section 11, and our problem now is to put them together and to have them interacting. For that purpose we need the total Hamiltonian

$$H_T = H_E + H_F + H_Q, \tag{16.3}$$

where H_E is the Hamiltonian for the electrons and positrons alone, H_F the Hamiltonian for the electromagnetic field alone, and H_Q the interaction energy. H_E is a function of ψ and $\bar{\psi}$ alone, and H_F is a function of a A_μ, B_μ, the Hamiltonian variables describing the electromagnetic field. The interaction energy H_Q is a function of all the variables ψ, $\bar{\psi}$, A_μ, B_μ.

Our problem now is to decide what H_Q shall be. In the classical theory we considered the case when the interaction is described by a term in the action integral involving the potentials only and no derivatives of the potentials, and we found for H_Q the expression (16.2) if j_μ is independent of the A's and of the velocities ${q_n}'^0$. These conditions for j_μ are satisfied by (11.9), which gives

$$H_Q = -e\int A^\mu \bar{\psi}\alpha_\mu \psi \, d^3x. \tag{16.4}$$

Let us take over this expression as it stands into the quantum theory.

The classical theory is only to be used as a hint for getting a quantum theory, and we have to check and see that we really have a relativistic quantum theory when we work with the Hamiltonian (16.3) with H_Q given by (16.4). We have to check whether the field equations to which it leads are relativistic, and we must check whether the commutation and anticommutation relations to which it leads are relativistic. Now the question of whether the commutation and anticommutation relations are relativistic is something we discussed in a general way in Section 13, and we found that when the interaction Hamiltonian does not involve derivatives of the dynamical variables, it does not disturb the commutation and anticommutation relations. Well, there are no derivatives occurring in expression (16.4) for H_Q, so this is the simple kind of intersection which does not disturb commutation and anticommutation relations and it is quite all right from that point of view.

We still have to check whether our field equations are satisfactory. We work them out from Heisenberg's equations of motion. Let's take all the different variables we have in our problem, and see what equations of motion we get for them. Start with A_μ. A_μ commutes with $H_E + H_Q$, so

$$\frac{\partial A_\mu}{\partial x_0} = [A_\mu, H_T] = [A_\mu, H_F] = B_\mu,$$

which is the same formula that we have for the electromagnetic field alone. The B_μ's retain the meaning of being the time derivatives of the A_μ's. The interaction does not disturb that relationship.

Now let us form

$$\frac{\partial B_\mu}{\partial x_0} = [B_\mu, H_T] = [B_\mu, H_F] + [B_\mu, H_Q];$$

the electron part of the energy does not contribute anything. The term $[B_\mu, H_F]$ is the same as with no interaction and is thus $A_\mu{}^{,rr}$. To calculate $[B_\mu, H_Q]$ we see that B_μ commutes with j_μ and that

$$[A_{\mu x}, B_{\nu x'}] = -4\pi g_{\mu\nu}\, \delta_3(x - x'),$$

so $[B_\mu, H_Q] = 4\pi j_\mu$ and $\partial B_\mu/\partial x_0 = A_\mu{}^{,rr} + 4\pi j_\mu$. Now when you combine this equation with our previous $B_\mu = \partial A_\mu/\partial x_0$ you get

$$\frac{\partial^2 A_\mu}{\partial {x_0}^2} = A_\mu{}^{,rr} + 4\pi j_\mu, \tag{16.5}$$

the Maxwell equation for the electromagnetic field produced by the current density j_μ.

Now let's take one of the components of ψ and write the Heisenberg equation of motion:

$$i\hbar \frac{\partial}{\partial x_0}\psi_{ax} = i\hbar[\psi_{ax}, H_T] = [\psi_{ax}, H_E + H_Q]_- .$$

We have to compute the commutator now. To do that we multiply $H_E + H_Q$ by ψ_{ax} first on the left, then on the right, and subtract. Now

$$H_E + H_Q = \int \bar{\psi}_b[\alpha_r(-i\hbar\psi^{,r} + eA_r\psi) + m\alpha_m\psi - eA_0\psi]_b \, d^3x,$$

so we get

$$\begin{aligned} i\hbar \frac{\partial}{\partial x_0}\psi_{ax} &= \int [\psi_a, \bar{\psi}_b{}']_+[\alpha_r(-i\hbar\psi'^{,r'} + eA_r{}'\psi') \\ &\qquad + m\alpha_m\psi' - eA_0{}'\psi']_b \, d^3x' \\ &= \int \delta_{ab}\,\delta_3(x - x')[\alpha_r(-i\hbar\psi'^{,r'} + eA_r{}'\psi') \\ &\qquad + m\alpha_m\psi' - eA_0{}'\psi']_b \, d^3x', \end{aligned}$$

and the result is

$$i\hbar \frac{\partial}{\partial x_0}\psi_{ax} = [\alpha_r(-i\hbar\psi^{,r} + eA_r\psi) + m\alpha_m\psi - eA_0\psi]_a .$$

This equation is precisely the ordinary one-electron equation involving the q-numbers A_μ for the field. We know from the one-electron theory that this equation remains invariant under Lorentz transformations. Therefore, the Heisenberg equation for ψ_{ax} is all right. The equation for $\bar{\psi}$ must be just the conjugate complex equation, and that must also be all right.

So all the field equations that follow from the total Hamiltonian (16.3) are relativistic, as one would expect from classical analogy.

The interaction energy given by the classical theory is apparently the correct one to use.

The Hamiltonian by itself does not give a complete theory; to get a complete theory we must bring in the constraints. For the case of no charges we had the constraints $\partial A_\mu/\partial x_\mu \approx 0$. We must determine if they are still all right or if they get disturbed by the interaction. The question of whether they are all right or not is the question of whether they are consistent or not.

In the case of no interaction we checked the consistency in two ways, one of them consisting of taking them at two general points in space-time and working out the Poisson bracket, from (15.14); we found that it vanishes. Well, we can't do that now; we can't take the $\partial A_\mu/\partial x_\mu$ at two different times and compare them, because that would involve integrating the Heisenberg equations of motion, which is a very complicated problem. We have to check the consistency conditions without integrating the Heisenberg equations of motion and must use the other method, which consists of taking all the constraints that hold at one time and seeing whether they are consistent with the equations of motion and with one another.

At a particular time we have

$$\frac{\partial A_\mu}{\partial x_\mu} \approx 0, \qquad \frac{\partial}{\partial x_0}\,\frac{\partial A_\mu}{\partial x_\mu} \approx 0, \qquad \frac{\partial^2}{\partial x_0{}^2}\,\frac{\partial A_\mu}{\partial x_\mu} \approx 0,$$

and so on. Now if you look at these equations you see that, as in the case of no charges, only the first two are independent when you take into account the field equations (16.5). The third one is a consequence of the first two, because it gives

$$A_\mu{}^{,\mu rr} + 4\pi j_\mu{}^{,\mu} \approx 0.$$

Now $j_\mu{}^{,\mu}$ vanishes because of conservation of charge. To prove it in the present theory we refer to the one-electron theory. We know that the charge is conserved in the one-electron theory. From that we can deduce the conservation of charge in the present theory, because the field equations for ψ in the present theory are of exactly the same form as in the one-electron theory. Just taking over the one-electron theory and making the potentials into q-numbers instead of c-numbers does not in any way affect the proof of the conservation of charge.

Well we are left with $A_\mu{}^{,\mu rr} \approx 0$, which is valid as a consequence of $A_\mu{}^{,\mu} \approx 0$ holding at all points in three-dimensional space. The result is that when the field equations are used, the constraint $A_{\mu,}{}^{\mu 00} \approx 0$ doesn't give us anything new. Similarly, the higher derivatives with respect to x_0 don't give us anything new. The fact that these higher derivatives don't give us anything new is the check that the constraints are consistent with the field equations, and thus that they commute with the Hamiltonian.

We are left with two constraint equations: each of them is a threefold infinity of conditions, because each of them holds for all points x_1, x_2, x_3. We let us write them out in terms of our Hamiltonian variables and check the Poisson-bracket relationships. The first gives us

$$B_0 + A_{r,}{}^{r} = \phi_1 \approx 0, \tag{16.6}$$

as in (15.10). The second gives $\partial^2 A_0/\partial x_0{}^2 + B_{r,}{}^{r} \approx 0$. Now, using the field equations, we replace the second time derivative by $A_{0,}{}^{rr} + 4\pi j_0$ and write the constraint

$$(A_{0,}{}^{r} + B_r)_{,}{}^{r} + 4\pi j_0 = \phi_2 \approx 0, \tag{16.7}$$

like (15.10) with the extra term $4\pi j_0$. You will recognize that equation as $-\operatorname{div} \mathcal{E} + 4\pi j_0 \approx 0$, one of the Maxwell's equation for $\mathcal{E}$, the electric field.

To check whether these constraints are consistent, we have to check whether the equations (15.11) still hold with the extra term $4\pi j_0$ in ϕ_2. The first two are obvious. For the third we have

$$[\phi_2, \phi_2'] = [(A_{0,}{}^{r} + B_r)_{,}{}^{r} + 4\pi j_0, \qquad (A_0'{}_{,}{}^{s'} + B_s')_{,}{}^{s'} + 4\pi j_0'].$$

We get one new term $16\pi^2[j_0, j_0']$. So our work for the case of no charges tells us that everything is all right, except for this new term involving the charge density at two different points. Now what is this Poisson bracket? There is no need to bother to work it out in detail, because we can see on general grounds that it has to vanish. The anticommutator of ψ and $\bar{\psi}'$, in which they are not differentiated, involves only the delta function: so $[j_0, j_0']$ = some coefficient times $\delta_3(x - x')$. Now $[j_0, j_0']$ has to be antisymmetrical between x and x'. Any multiple of the δ-function is symmetrical unless it is zero; hence

$$[j_0, j_0'] = 0.$$

So we have checked that the constraints commute weakly with the Hamiltonian and with each other; actually they commute strongly with each other, as in the case of no charges. So we can still give a meaning to the reality of all q-numbers, not merely the physical ones.

There are two important applications of this theory of quantum electrodynamics: the anomalous magnetic moment of the electron and the Lamb shift. These are applications that one cannot make with a more primitive theory. One needs the full quantum electrodynamics to get the results. I shall now show how the calculations of the anomalous magnetic moment and the Lamb shift are to be made by working from the Hamiltonian (16.3) and (16.4).

We have to solve the equations of motion, and as I have discussed before, it is no use trying to solve the Schrödinger equations of motion. We might set up formally a ket vector satisfying the Schrödinger equation, but that ket vector would not stay in Hilbert space. The only way to carry on with this work is to confine our attention to the Heisenberg picture and obtain solutions of the Heisenberg equations of motion with the Hamiltonian (16.3) and (16.4).

§17. *Solution of the Heisenberg Equations of Motion: General*

Now let us first discuss in general terms what solutions of the Heisenberg equations will look like. If we have any Hamiltonian H, we can take the Heisenberg equation for a dynamical variable ξ, $i\hbar(d\xi/dt) = \xi H - H\xi$, and try to get a solution of it. That solution would be in the form $\xi = f(t$, constants of the motion). These constants of the motion would be q-numbers, noncommuting, of course. What should we take them to be?

Well, we may take them to be the dynamical variables at one particular time, so as to get

$$\xi = f\big(t, q(t_0), p(t_0)\big).$$

We might be given all the field quantities at time t_0 and calculate the field quantities at a general time in terms of them. I call this method 1A.

Of course, you might say that from the relativistic point of view this is not very good. It means taking ξ, a field quantity at a general point in space-time, and expressing it in terms of field quantities over all positions of space at one particular time. That is a very unrelativistic way of proceeding.

Alternatively, we might express ξ as a function of the four x's and of the ingoing fields, to get $\xi_{x_0x_1x_2x_3} = f(x_\mu$, ingoing fields). This I call method 1B. How are ingoing fields defined? We define them as in the classical theory. Consider the electromagnetic field, for instance. We have the equation

$$\Box A_\mu = 4\pi j_\mu,$$

holding in the quantum theory as in the classical theory. We can solve this equation in the quantum theory as in the classical theory and express $A_\mu = A_{\mu\,\mathrm{in}} + A_{\mu\,\mathrm{ret}}$, where $A_{\mu\,\mathrm{ret}}$ is defined in terms of j_μ by the classical formula. The noncommutation of the j_μ's at different places doesn't matter here because everything is linear in the j_μ's, and $A_{\mu\,\mathrm{ret}}$ appears as the usual linear function of the j_μ's. We then define $A_{\mu\,\mathrm{in}} = A_\mu - A_{\mu\,\mathrm{ret}}$. We then have $\Box A_{\mu\,\mathrm{in}} = 0$.

That is how one defines the ingoing field in the electromagnetic case and one can do a corresponding thing for the ψ fields. These definitions go just the same as in the classical theory, because they involve only linear relationships.

You might try to solve the Heisenberg equations of motion by getting each field quantity at a particular point in space-time expressed in terms of the coordinates at that point in space-time and the ingoing fields or the Fourier coefficients of the ingoing fields. You might say that this is a nice method to work with because it is relativistic, and it would enable one to set up the whole theory in a manifestly covariant form. However, the ingoing field is a rather complicated concept when one starts from the Heisenberg equations of motion. Its existence depends on the convergence of the integral for the retarded field—an integral that goes back to the infinite past. There are examples for which the concept fails. A further discussion of this question will be given in Section 32.

It is not desirable to have the theory depending at this early stage in its development on such a shaky concept as the ingoing field. For this reason one should reject method 1B and fall back on 1A. We can improve on method 1A by adopting a different method, which I call method 2.

Method 2: Find a quantity K involving t explicitly such that it is a constant of the motion. Express K as a function of t explicitly and the q's and the p's, the dynamical variables, at time t. The advantage of method 2 compared with 1A is that there is only one time t involved.

The quantity K involves the time explicitly and so it doesn't satisfy the same equation of motion that ξ satisfies. The equation for ξ is a valid equation only for dynamical variables and functions of dynamical variables that do not involve the time explicitly. K does involve the time explicitly, so it must satisfy the more general Heisenberg equation

$$i\hbar\frac{dK}{dt} = KH - HK + i\hbar\frac{\partial K}{\partial t}\cdot$$

The extra term comes from the explicit dependence of K on t.

The requirement that K is constant means $\partial K/\partial t = 0$ or

$$i\hbar\frac{\partial K}{\partial t} = HK - KH. \tag{17.1}$$

We get this new equation of motion to work with. You see that there is the opposite sign coming in, and also the total differentiation with respect to time is replaced by partial differentiation. You must remember that in this partial differentiation q_t and p_t count as constants; i.e., $\partial q_t/\partial t = 0$, $\partial p_t/\partial t = 0$. q_t and p_t have total derivatives with respect to time but no partial derivatives.

Method 2 is the method with which we are going to work.

We now have to consider the problem of getting solutions of (17.1) with H equal to the Hamiltonian for quantum electrodynamics. We shall use a perturbation method which assumes that H_Q is small. H_Q involves the parameter e (the charge of the electron), and that is small compared with the quantity of the same dimensions constructed with the Planck constant. On the basis of e being small, we can use this perturbation method and express everything as a power series in e, and we just work with the early terms in that power series.

The first thing we do is pass over to the interaction representation. I suppose you are quite familiar with the interaction representation as used in connection with the Schrödinger picture. One can use a corresponding interaction representation in connection with the Heisenberg picture, and it goes like this.

Let us write $H = H_0 + H_Q$, where H_0 is the proper energy. For any q-number α, define $\alpha^* = e^{iH_0t/\hbar}\alpha e^{-iH_0t/\hbar}$. This is just an ordinary unitary transformation, because H_0 can be considered as an operator in Hilbert space.

Let us apply that transformation to the quantity K and make a partial differentiation with respect to t:

$$\begin{aligned} i\hbar\,\frac{\partial K^*}{\partial t} &= e^{iH_0t/\hbar}\left(-H_0K + i\hbar\frac{\partial K}{\partial t} + KH_0\right)e^{-iH_0t/\hbar} \\ &= e^{iH_0t/\hbar}(-H_0K + HK - KH + KH_0)e^{-iH_0t/\hbar} \\ &= e^{iH_0t/\hbar}(H_QK - KH_Q)e^{-iH_0t/\hbar} = H_Q{}^*K^* - K^*H_Q{}^*. \end{aligned} \tag{17.2}$$

This is the equation of motion for K in the interaction representation. It achieves the usual result which one achieves by going over to the interaction representation: We get equations of motion in which things vary only on account of the interaction; the proper part of the energy doesn't contribute anything to the variation.

This passage to the interaction representation is exact. We are now going to make use of the assumption that the interaction energy is small and express $K^* = K_0{}^* + K_1{}^* + K_2{}^* + \cdots$, the various terms being of different orders of smallness in e. Now we substitute this expression for K^* in the equation of motion (17.2) and equate terms of corresponding orders of smallness. We get

$$i\hbar \frac{\partial K_0{}^*}{\partial t} = 0, \tag{17.3}$$

$$i\hbar \frac{\partial K_1{}^*}{\partial t} = H_Q{}^* K_0{}^* - K_0{}^* H_Q{}^*, \tag{17.4}$$

$$i\hbar \frac{\partial K_2{}^*}{\partial t} = H_Q{}^* K_1{}^* - K_1{}^* H_Q{}^*, \tag{17.5}$$

and so on.

Now what do these equations mean? First of all, (17.3) shows that $K_0{}^*$ must not involve the time explicitly. So $K_0{}^*$ equals a function of the q_t's and p_t's, the dynamical variables at time t. We choose a suitable $K_0{}^*$ to start off the calculation. Then we proceed to get $K_1{}^*$ by evaluating the $[K_0{}^*, H_Q{}^*]_-$ commutator and making a time integration. We proceed to get $K_2{}^*$ by evaluating the $[K_1{}^*, H_Q{}^*]_-$ commutator and making another time integration, and so on. In that way we get a solution of the Heisenberg equations of motion by this step-by-step method.

When we have got these K^* quantities we can go back to the K quantities by just inverting the unitary transformation, and then we have our solution. One point I want to emphasize is that $K_0{}^*$ involves only dynamical variables and does not involve the time explicitly. When you go back to K_0 you will get something that does involve the time explicitly.

§18. *Solution of the Heisenberg Equations of Motion: Quantum Electrodynamics*

We shall be concerned with the one-electron problem in the calculation of the anomalous magnetic moment and the Lamb shift, and for that purpose we shall take $K_0{}^*$ to be the operator of the emission of an electron into a certain state.

It will be necessary in these calculations to suppose the existence of a static magnetic or electric field. To calculate the anomalous magnetic moment, we suppose that there is a static magnetic field, and then we examine how this static magnetic field influences the energy of the electron that is being emitted. Similarly, for the Lamb shift we suppose that there is a static electric field and we determine how that static electric field influences the energy of the electron that is being emitted.

In our derivation of the Hamiltonian we didn't consider static fields, but we can bring them in in a trivial way. A static field is described by potentials $\mathcal{A}_\mu$ which don't vary with the time; they are not dynamical variables; they are just c-numbers, functions of x_1, x_2, x_3. So to bring in the static fields we add on these $\mathcal{A}$'s to the other A's. Thereby we bring in some extra terms to the Hamiltonian.

Let us simplify future work by taking $h = 1$ as well as $c = 1$. When one gets final results, one can fill them in to make the dimensions right.

We shall consider how to introduce the absorption and emission operators which are going to be our working variables. They must be defined as dynamical variables that do not involve the time explicitly.

Let us take first the electromagnetic field variables. For them the work is the same as in Section 15 for the case of no charges. We start with $A_{\mu x}$ and $B_{\mu x} = \partial A_{\mu x}/\partial x_0$ and make a Fourier resolution like (9.4) and (9.5). It reads

$$\begin{aligned} A_{\mu x} &= \int [A_{\mu k} e^{-i(\mathbf{k},\mathbf{x})} + \overline{A}_{\mu k} e^{i(\mathbf{k},\mathbf{x})}]\, d^3k, \\ B_{\mu x} &= i \int |k| [A_{\mu k} e^{-i(\mathbf{k},\mathbf{x})} - \overline{A}_{\mu k} e^{i(\mathbf{k},\mathbf{x})}]\, d^3k. \end{aligned} \tag{18.1}$$

These equations define $A_{\mu k}$ and $\overline{A}_{\mu k}$. We just have to invert the Fourier resolutions and we get two equations which determine $A_{\mu k}$ and $\overline{A}_{\mu k}$.

What I'd like to emphasize about these equations is that there's no time t occurring in them anywhere. The equations that define $A_{\mu k}$ and $\overline{A}_{\mu k}$ don't involve the time and so we have $(\partial/\partial t)A_{\mu k} = 0$, $\partial \overline{A}_{\mu k}/\partial t = 0$. This way of introducing Fourier coefficients gives Fourier coefficients that are not explicitly dependent on the time. Of course, the total $(d/dt)A_k$ is something which is not zero.

We know what that total dependence on the time is in the case of no interaction. For no interaction $A_{\mu k}$ is proportional to $e^{i|\mathbf{k}|t}$, so we then have $(d/dt)A_{\mu k} = i|\mathbf{k}|A_{\mu k}$. This means

$$[A_{\mu k}, H_0] = i|\mathbf{k}|A_{\mu k}. \tag{18.2}$$

This equation must also hold when there is interaction. It just tells us something about the commutation relationship of $A_{\mu k}$ with H_0, which must hold independent of whether there is interaction or not.

Alternatively, one could obtain (18.2) by working out the Poisson bracket directly. We have

$$[A_{\mu k}, A_{\lambda k'}]_- = 0$$

and

$$[\overline{A}_{\mu k}, A_{\lambda k'}]_- = -\frac{g_{\mu\lambda}}{4\pi^2|\mathbf{k}|}\,\delta_3(\mathbf{k} - \mathbf{k}') \tag{18.3}$$

from our work on the electromagnetic field by itself, and the commutation relations are not disturbed by the interaction. Equation (18.2) is a consequence of the commutation relations (18.3) and therefore holds also when there is interaction.

Equation (18.2) can be written

$$A_k H_0 = (H_0 - |\mathbf{k}|)A_k.$$

We can generalize that to

$$A_k f(H_0) = f(H_0 - |\mathbf{k}|)A_k$$

for any function f; it holds certainly for a power series, and we don't need to apply it to other functions. We can use this now to calculate

$$A_k{}^* = e^{iH_0t}A_k e^{-iH_0t} = e^{iH_0t}e^{-i(H_0-|\mathbf{k}|)t}A_k = e^{i|\mathbf{k}|t}A_k.$$

There we have calculated the starred field variables that occur in the interaction representation.

To introduce the absorption and emission operators for the electrons and positrons, we must take into account the static electric and magnetic fields. They give a further term in the Hamiltonian: $\int \mathcal{A}^\mu j_\mu \, d^3x$. This further term is going to be put in H_0 and not H_Q. The perturbation is going to remain the same as before when we had no $\mathcal{A}_\mu$ field.

With the term $\int \mathcal{A}^\mu j_\mu \, d^3x$ included in H_0, we shall be wanting to use the wave functions for the one-electron problem in the static field. Let's suppose that these functions have been worked out. Finding the different eigenfunctions of the energy for one electron in the static field is just a trivial problem, without any field quantization involved.

In this one-electron problem let E_n be an eigenvalue of the energy and $\langle xa|n\rangle$ the corresponding eigenfunction. The eigenfunction is a function of the three x's and a function of a; a takes on the four values, corresponding to the four components of the wave function. This n is just a symbol to remind us of which eigenfunction we are talking about. I'm going to write down the equations as though the energy spectrum is discrete; if it is not discrete, it just means the usual change in the way of writing, with the sums over the discrete n values replaced by integrals.

I would like to write down now the wave equation which the eigenfunction satisfies. We have as the energy operator for one electron in the static field,

$$\alpha_r\left(-i\frac{\partial}{\partial x_r} + e\mathcal{A}_r\right) + \alpha_m m - e\mathcal{A}_0 . \tag{18.4}$$

We take the ab matrix element of (18.4) and multiply it into $\langle xb|n\rangle$. We get

$$\left[\alpha_r\left(-i\frac{\partial}{\partial x_r} + e\mathcal{A}_r\right) + \alpha_m m - e\mathcal{A}_0\right]_{ab}\langle xb|n\rangle = E_n\langle xa|n\rangle.$$

This equation is the usual equation for determining eigenfunctions of the operator (18.4). I am going to suppose that we have worked out all these eigenfunctions and we've got them all normalized, so that they satisfy the usual orthonormal relations:

$$\int\langle n|xa\rangle\langle xa|n'\rangle\, d^3x = \delta_{nn'}, \tag{18.5}$$

a summation being implied over the four values of a; and

$$\sum_n \langle xa|n\rangle\langle n|x'b\rangle = \delta_{ab}\,\delta_3(x - x');$$

with $\langle n|xa\rangle$ the conjugate complex of $\langle xa|n\rangle\eta$.

Now let us define $\psi_n = \int\langle n|x_a\rangle\psi_{ax}\, d^3x$, a summation being implied over a; or we can invert that equation and write $\psi_{ax} = \sum_n\langle xa|n\rangle\psi_n$. The work with eigenfunctions previously has just involved c-numbers, and now I'm introducing q-numbers. These ψ_{ax}'s are q-numbers, and

I'm defining these new q-numbers ψ_n. We shall be using also the conjugate complexes

$$\bar{\psi}_n = \int \bar{\psi}_{ax}\langle xa|n\rangle\, d^3x, \qquad \bar{\psi}_{ax} = \sum_n \bar{\psi}_n\langle n|xa\rangle.$$

Here we introduce the q-numbers ψ_n, $\bar{\psi}_n$; what are their anticommutation relationships? One very easily finds that

$$[\psi_n, \psi_n']_+ = 0, \qquad [\psi_n, \bar{\psi}_{n'}]_+ = \delta_{nn'}.$$

Now we started off with ψ_{ax}, and that is something which does not involve the time explicitly: $(\partial/\partial t)\psi_{ax} = 0$, $(\partial/\partial t)\bar{\psi}_{ax} = 0$. The equations that introduce ψ_n, $\bar{\psi}_n$ do not involve the time explicitly; so we can infer that $\partial\psi_n/\partial t = 0$, $\partial\bar{\psi}_n/\partial t = 0$.

Now H_0, the proper energy, is equal to $H_E + E_F$, the electron part and the electromagnetic field part. H_E, the electron part, is

$$H_E = \int \bar{\psi}_{ax}\left[\alpha_r\left(-i\frac{\partial}{\partial x_r} + e\mathcal{A}_r\right) + \alpha_m m - e\mathcal{A}_0\right]_{ab}\psi_{bx}\, d^3x.$$

When we go over to these new ψ_n, $\bar{\psi}_n$ variables, we get

$$H_E = \sum_{nn'}\int \bar{\psi}_n\langle n|xa\rangle\left[\alpha_r\left(-i\frac{\partial}{\partial x_r} + e\mathcal{A}_r\right) + \alpha_m m - e\mathcal{A}_0\right]_{ab} \times \langle xb'|n'\rangle\psi_{n'}\, d^3x.$$

Now we make use of the equation the eigenfunction satisfies, so that we get

$$H_E = \sum_{nn'}\int \bar{\psi}_n\langle n|xa\rangle E_{n'}\langle xa|n'\rangle\psi_{n'}\, d^3x.$$

Now the only things involving x here are the eigenfunctions, and we can make use of the relation (18.5) and get

$$H_E = \sum_{nn'} \bar{\psi}_n E_{n'}\,\delta_{nn'}\psi_{n'} = \sum_n E_n\bar{\psi}_n\psi_n. \tag{18.6}$$

This is what you would expect the energy of the electrons to be, referred to the ψ_n, $\bar{\psi}_n$ variables.

The variables ψ_n and $\bar{\psi}_n$ are the operators of emission and of absorption into the different states n. For E_n greater than 0, ψ_n is the operator of absorption of an electron and $\bar{\psi}_n$ is the emission of an electron; and for E_n less than 0, ψ_n is emission of a positron and $\bar{\psi}_n$ is the absorption of a positron. That is the physical meaning of these ψ_n, $\bar{\psi}_n$ quantities.

Now we must do a piece of work corresponding to what we did in the case of the electromagnetic field variables, to find the starred quantities in the interaction representation. Again we can get the results simply by considering the case of no interaction, when

$$i\frac{d\psi_n}{dt} = E_n\psi_n .$$

So $[\psi_n, H_E]_- = E_n\psi_n$. This equation holds also when there is interaction. We can deduce from it $\psi_n H_0 = (H_0 + E_n)\psi_n$; and more generally $\psi_n f(H_0) = f(H_0 + E_n)\psi_n$, so

$$\psi_n{}^* = e^{iH_0t}\psi_n e^{-iH_0t} = e^{-iE_nt}\psi_n \qquad \text{and} \qquad \bar{\psi}_n{}^* = e^{iE_nt}\bar{\psi}_n .$$

I'm going to use an abbreviated notation for the Fourier resolution of the electromagnetic field,

$$A_{\mu x} = 2\int A_{\mu k}^{(\nu)} e^{-i(\mathbf{k},\mathbf{x})} d^3k. \tag{18.7}$$

I've introduced this new index ν, which takes on the values ± 1. For $\nu = 1$, we define $A_{\mu k}^{(1)} = A_{\mu k}$; for $\nu = -1$ we define $A_{\mu k}^{(-1)} = \bar{A}_{\mu(-k)}$. Furthermore, an average over both ν-values is implied, going with the integration over k. It is more convenient to work with an average over both values of ν and not a sum, because if ν drops out of a term, you can forget all about ν. The point of this notation is that we can write the Fourier expansion of $A_{\mu x}$ as a single term instead of two terms. You might say that it is not a very great saving in writing if we put down just one term instead of two—but we shall have to deal with quadratic functions of the potentials and then this notation will enable us to write down one term instead of four, and that's a considerable saving.

In terms of the absorption and emission operators, H_Q appears as

$$H_Q = -2e\int A_{\mu k}^{(\nu)}\bar{\psi}_x\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}\psi_x\, d^3x\, d^3k = -2e\sum_{nn'}\int A_{\mu k}^{(\nu)}\, d^3k\bar{\psi}_n \times \langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|n'\rangle\psi_{n'}.$$

Here I've introduced a new kind of quantity, $\langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|n'\rangle$, which is just a matrix element for the one-electron problem, defined by

$$\int \langle n|ax\rangle \alpha^{\mu}_{ab} e^{-i(\mathbf{k},\mathbf{x})} \langle bx|n'\rangle \, d^3x.$$

We now need $H_Q{}^*$, to appear in our perturbation theory:

$$\begin{aligned} H_Q{}^* &= -2e \sum_{nn'} \int \langle n|\alpha^{\mu} e^{-i(\mathbf{k},\mathbf{x})}|n'\rangle \, d^3k A^{(\nu)*}_{\mu k} \bar{\psi}_n{}^* \psi_{n'}{}^* \\ &= -2e \sum_{nn'} \int \langle n|\alpha^{\mu} e^{-i(\mathbf{k},\mathbf{x})}|n'\rangle \, d^3k A^{(\nu)}_{\mu k} \bar{\psi}_n \psi_{n'} e^{i(\nu|\mathbf{k}|+E_n-E_{n'})t}. \end{aligned}$$

I have used there the formula $A^{(\nu)*}_{\mu k} = A^{(\nu)}_{\mu k} e^{i\nu|\mathbf{k}|t}$. You see, when ν is equal to $+1$, we need the factor $e^{i|\mathbf{k}|t}$; when ν is equal to -1, we have the conjugate complex thing and we need the factor $e^{-i|\mathbf{k}|t}$; so if we put $e^{i\nu|\mathbf{k}|t}$, it's correct for both values of ν.

Now we are interested in a one-electron problem. We therefore take our $K_0{}^*$ to be the operator of emission of one electron into a positive energy state. Let us take $K_0{}^* = \bar{\psi}_i$, the operator of emission of the electron into the initial state i. We assume the energy of state i is positive; otherwise $\bar{\psi}_i$ would be the absorption operator of a positron instead of the emission operator of an electron. This choice of $K_0{}^*$ makes $\partial K_0{}^*/\partial t = 0$, so it is suitable for starting off the perturbation calculation.

We now proceed to find $K_1{}^*$:

$$\begin{aligned} i\frac{\partial K_1{}^*}{\partial t} = [H_Q{}^*, K_0{}^*]_- &= [H_Q{}^*, \bar{\psi}_i]_- \\ &= -2e \sum_{nn'} \int \langle n|\alpha^{\mu} e^{-ik\cdot x}|n'\rangle \, d^3k A^{(\nu)}_{\mu k} \\ &\quad \times [\bar{\psi}_n \psi_{n'}, \bar{\psi}_i]_- e^{i(\nu|\mathbf{k}|+E_n-E_{n'})t}. \end{aligned}$$

Now

$$\begin{aligned} [\bar{\psi}_n \psi_{n'}, \bar{\psi}_i]_- = \bar{\psi}_n \psi_{n'} \bar{\psi}_i - \bar{\psi}_i \bar{\psi}_n \psi_{n'} &= \bar{\psi}_n (\psi_{n'} \bar{\psi}_i + \bar{\psi}_i \psi_{n'}) \\ &= \bar{\psi}_n \, \delta_{n'i}. \end{aligned}$$

We have $\delta_{n'i}$ coming in, so we substitute i for n' and drop out the summation with respect to n'. We get

$$i\frac{\partial K_1{}^*}{\partial t} = -2e \sum_{n} \int \langle n|\alpha^{\mu} e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \, d^3k A^{(\nu)}_{\mu k} \bar{\psi}_n e^{i(\nu|\mathbf{k}|+E_n-E_i)t}.$$

Now we can carry out integration with respect to t; everything is constant except the exponential factor, so the integration is trivial:

$$K_1{}^* = 2e \sum_n \int \langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \, d^3k A_{\mu k}^{(\nu)} \bar{\psi}_n \frac{e^{i(\nu|\mathbf{k}|+E_n-E_i)t}}{\nu|\mathbf{k}| + E_n - E_i}.$$

Now we must go on to $K_2{}^*$. We have

$$\begin{aligned} i(\partial K_2{}^*/\partial t) &= [H_Q{}^*, K_1{}^*]_- \\ &= -4e^2 \sum_{nn'n''} \int \langle n'|\alpha^\lambda e^{-i(\mathbf{k}',\mathbf{x})}|n''\rangle\langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \, d^3k \, d^3k' \\ &\quad \times e^{i(\nu'\mathbf{k}|'|+E_{n'}-E_{n''})t} \frac{e^{i(\nu|\mathbf{k}|+E_n-E_i)t}}{\nu|\mathbf{k}| + E_n - E_i} \\ &\quad \times [A_{\lambda k'}^{(\nu')} \bar{\psi}_{n'}\psi_{n''}, A_{\mu k}^{(\nu)} \bar{\psi}_n]_-. \end{aligned} \tag{18.8}$$

All we have to do now is a time integration to get $K_2{}^*$. That would give us the solution of the Heisenberg equations of motion to the second order.

§19. *Normal Ordering*

Now this solution involves, of course, q-numbers, and would take different forms according to the order in which we arrange those q-numbers. We need to write this solution in a certain standard way and for that purpose we introduce a normal ordering of the q-numbers. We take it to be the ordering in which all the emission operators are to the left, all the absorption operators to the right. I take over this normal ordering from the usual work in the Schrödinger picture and assume it is physically significant. Some justification for it will be given in Section 31.

We must now, therefore, arrange the q-numbers in (18.8) so as to put all the emission operators to the left, all the absorption operators to the right. This rearranging of the factors, to get them into the normal order, will bring in some extra terms, and we must see just what they are. It does not matter about the order in which different absorption operators appear, or the order in which different emission operators appear; changing that order would at most involve a change of sign when we are dealing with fermion operators, and such a change does not bring in any extra terms; that change is to be looked upon as trivial.

The absorption operators are ψ_n for E_n greater than 0, $\bar{\psi}_n$ for E_n less than 0. Now what about the electromagnetic field variables? In the case of the transverse variables, we have operators of emission and absorption of photons, and there it is quite clear what is to be an emission operator and what an absorption operator. We get the $\overline{A}_k$ as absorption operators. How about the longitudinal variables? Well, there it is not clear what are emission operators and what are absorption operators. We shall again take $\overline{A}_k$ as an absorption operator. This should be looked upon as a definition of absorption operators in the case of the longitudinal variables. It is a convenient definition to work with, and makes it clear what the normal ordering is in all cases.

Question: Could you expand your explanation for the need of the normal ordering?

Answer: We want to get some numerical coefficients to which we can give a physical meaning. Now, you do not get any definite numerical coefficients at all if you leave the ordering unspecified, because changing the order changes the numerical coefficients. A normal ordering is for the purpose of getting well-defined numerical coefficients, which you have to do when you want to get something that is to be compared with experiment. q-numbers are not something you can compare directly with the experiment; you must take out numerical coefficients, and the only way to get numerical coefficients is to have some definite system of normal ordering. It is, of course, an assumption that this normal ordering is physically significant. I shall discuss this physical assumption later, in connection with the interpretation of the theory.

We adopt this definition of normal ordering and we assume that it is physically significant. We must now see what extra terms arise when we do the normal ordering.

We have the commutation relations (18.3) for the electromagnetic field quantities. Consider the product $A_{\mu k}^{(\nu)} A_{\lambda k'}^{(\nu')}$. We see that if ν and ν' are both equal to 1 or -1, this is already in the normal order. Also $A_{\mu k}^{(1)} A_{\lambda k'}^{(-1)}$ is in normal order. The only case in which we do not have normal ordering is where we have $\nu = -1$ on the left and $\nu' = 1$ on the right. And $A_{\mu k}^{(-1)} A_{\lambda k'}^{(1)}$ differs from the normal ordering of the factors by the commutator in the second of equations (18.3), so we get $A_{\mu k}^{(-1)} A_{\lambda k'}^{(1)} = \text{normal order} - g_{\mu\lambda}/4\pi^2|\mathbf{k}|\ \delta_3(k + k')$. We have to change $k - k'$ to $k + k'$ in the δ-function because of our definitions of $A_{\mu k}^{(\nu)}$ with various ν-values. We can write one formula that includes all these results,

$$A_{\mu k}^{(\nu)} A_{\lambda k'}^{(\nu')} = \text{normal order} - \frac{g_{\mu\lambda}}{4\pi^2|\mathbf{k}|}\,\delta_3(k + k')\left(\frac{1-\nu}{2}\right)\left(\frac{1+\nu'}{2}\right).$$

There are different ways of writing the ν-factors, because

$$(1 - \nu)(1 + \nu') = (1 - \nu)(\nu' - \nu) = (1 + \nu')(\nu' - \nu).$$

These are just algebraic formulas that hold because ν and ν' are able to take on only the values 1 and -1.

Now consider the product $\psi_n \bar{\psi}_{n'}$. This is in the normal order for all n if $E_{n'}$ is negative. In general we have

$$\psi_n \bar{\psi}_{n'} = \text{normal order} + \delta_{nn'} \Theta(E_{n'}).$$

The Θ-function is the step function, which is used quite a bit in field theory. It is defined by

$$\Theta(x) = 1 \quad \text{for} \quad x > 0, \qquad \Theta(x) = 0 \quad \text{for} \quad x < 0.$$

Alternatively, $\Theta(x) = \frac{1}{2}(1 + x/|x|)$. Note that

$$\Theta(x) + \Theta(-x) = 1, \qquad \Theta(x) - \Theta(-x) = x/|x|. \tag{19.1}$$

We can also write $\psi_n \bar{\psi}_{n'} = \text{normal order} + \delta_{nn'} \Theta(E_n)$, because nn' have to be the same for $\delta_{nn'}$ to contribute anything. Then we have the corresponding formula

$$\bar{\psi}_n \psi_{n'} = \text{normal order} + \delta_{nn'} \Theta(-E_n).$$

We must now take the commutator in (18.8) and put it in normal order. If we write it out in full it is like this:

$$A^{(\nu')}_{\lambda k'} A^{(\nu)}_{\mu k} \bar{\psi}_{n'} \psi_{n''} \bar{\psi}_n - A^{(\nu)}_{\mu k} A^{(\nu')}_{\lambda k'} \bar{\psi}_n \bar{\psi}_{n'} \psi_{n''}.$$

Putting it in normal order using the formulas for normal order given above, we get

$$\begin{aligned}
&- \frac{g_{\lambda\mu}}{4\pi^2|\mathbf{k}|} \delta_3(k + k') \tfrac{1}{4}(\nu - \nu') \\
&\times \{(1 + \nu)[\bar{\psi}_{n'}\, \delta_{nn''} \Theta(E_n) + \bar{\psi}_n\, \delta_{n'n''} \Theta(-E_{n'})] \\
&+ (1 - \nu)[\bar{\psi}_n\, \delta_{n'n''} \Theta(-E_{n'}) - \bar{\psi}_{n'}\, \delta_{nn''} \Theta(-E_n)]\} \\
&+ \text{extra terms},
\end{aligned}$$

the extra terms involving more than one emission or absorption operator.

We must substitute this commutator in (18.8), and that gives us

$$\begin{aligned} i\frac{\partial K_2{}^*}{\partial t} &= \frac{e^2}{\pi}\left\{\sum_{n'} \bar{\psi}_{n'} Y_{n'i} e^{i(E_{n'}-E_i)t} + \sum_n \bar{\psi}_n Z_{ni} e^{i(E_n-E_i)t}\right\} \\ &\quad + \text{extra terms} \\ &= \frac{e^2}{\pi}\sum_n \bar{\psi}_n\{Y_{ni} + Z_{ni}\}e^{i(E_n-E_i)t} + \text{extra terms}, \qquad (19.2) \end{aligned}$$

where the coefficients Y_{ni} and Z_{ni} have the following values:

$$\begin{aligned} Y_{n'i} = (4\pi)^{-1}\sum_n \int \langle n'|\alpha_\mu e^{i(\mathbf{k},\mathbf{x})}|n\rangle\langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \frac{\nu-\nu'}{\nu|\mathbf{k}| + E_n - E_i} \\ \times \{(1+\nu)\Theta(E_n) - (1-\nu)\Theta(-E_n)\}\frac{d^3k}{|k|} \qquad (19.3) \end{aligned}$$

and

$$\begin{aligned} Z_{ni} = (2\pi)^{-1}\sum_{n'} \Theta(-E_{n'})\int \langle n'|\alpha_\mu e^{i(\mathbf{k},\mathbf{x})}|n'\rangle\langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \\ \times \frac{\nu-\nu'}{\nu|\mathbf{k}| + E_n - E_i}\frac{d^3k}{|\mathbf{k}|}. \qquad (19.4) \end{aligned}$$

There is one immediate simplification we can make. We have to average over all values of ν'; that means that the terms containing ν' linearly drop out. Thus the $\nu - \nu'$ factor in the numerator, for both Y and Z, can be replaced by ν.

§20. *The Energy Change*

The extra terms in (19.2) all refer to more complicated physical processes with emission and absorption of more than one particle, whereas the Y and Z terms just mean a modification in our original emission operator produced by the perturbation. Suppose we pick out the one term in (19.2) for which n is equal to i. It is $(e^2/\pi)(Y_{ii} + Z_{ii})\bar{\psi}_i$. This term represents a change in the energy of the emitted particle.

Let us see just what energy change is represented by the coefficient of $\bar{\psi}_i$ here. The most definite way of getting the desired result is to consider the effect of another perturbation in the Hamiltonian. Consider a perturbation V of the form $V = \sum_n \bar{\psi}_n b_n \psi_n$ where b_n is some small c-number for each value of n. Let us suppose a perturbation like this to be added to our Hamiltonian. Such a perturbation will obviously cause a change in the emission energy of amount b_n, because, from (18.6), the part of our Hamiltonian that refers to the particles only is just $\sum_n \bar{\psi}_n E_n \psi_n$. Since E_n gets changed to $E_n + b_n$, it means that b_n is the small change in the energy of the emitted particle.

If the problem is handled by a perturbation method, this V gives $i(\partial K_1^*/\partial t) = V^* K_0^* - K_0^* V^*$. Taking $K_0^* = \bar{\psi}_i$, as before, $i(\partial K_1^*/\partial t)$ works out to be $i(\partial K_1^*/\partial t) = \bar{\psi}_i b_i$ to first order in the b's. This $\partial K_1^*/\partial t$ can be compared with (19.2), and it enables us to infer that the change in the energy of the emitted particle is precisely the term for $n = i$: $(e^2/\pi)(Y_{ii} + Z_{ii})$.

This is the expression that we need to evaluate to get the anomalous magnetic moment and the Lamb shift. We have the general formulas (19.3) and (19.4) for the matrix elements of Y and Z, and it is the diagonal elements that will be the important ones.

We can write Z in the form

$$Z_{ni} = \pi^{-1} \int J_k{}^{\mu} \frac{\langle n|\alpha_\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle}{\nu|\mathbf{k}| + E_n - E_i} \nu \frac{d^3k}{|\mathbf{k}|}, \tag{20.1}$$

where $J_k{}^{\mu} = \frac{1}{2}\sum_{n'}\Theta(-E_{n'})\langle n'|\alpha^{\mu} e^{i(\mathbf{k},\mathbf{x})}|n'\rangle$. Now $J_k{}^{\mu}$ can be written as a diagonal sum,

$$J_k{}^{\mu} = \tfrac{1}{2}\langle\langle \alpha^{\mu} e^{i(\mathbf{k},\mathbf{x})} \Theta(-E)\rangle\rangle, \tag{20.2}$$

where $\langle\langle\ \ \rangle\rangle$ means diagonal sum and E is the energy operator for one particle in the external field,

$$E = \alpha_r(p_r + e\mathcal{A}_r) + \alpha_m m - e\mathcal{A}_0.$$

E is not a q-number here, but it is a matrix for the one-electron problem. It is a matrix whose elements are all c-numbers. $\Theta(-E)$, of course, is $\frac{1}{2}(1 - E/|E|)$. How is $|E|$ defined? Well, this is the modulus of a one-electron operator, a fairly simple operator. We have $|E| = +(E^2)^{1/2}$. So $|E|$ is quite a well-defined operator, perhaps a bit awkward to work out; in fact, the only practical way to work it out in general is to use a representation in which E is diagonal.

The diagonal sum of a matrix is the same in all representations. The diagonal sum (20.2) which we find to give the correct expression for $J_k{}^{\mu}$ when referred to the representation with E diagonal, can be evaluated more conveniently in that representation in which the momentum variables are diagonal. Let us proceed to work in the representation in which the p's are diagonal. We have

$$\langle p'|e^{i(\mathbf{k},\mathbf{x})}|p''\rangle = \delta_3(p' - p'' - k). \tag{20.3}$$

This follows from $p_r e^{i(\mathbf{k},\mathbf{x})} = e^{i(\mathbf{k},\mathbf{x})}(p_r + k_r)$. So we get

$$\begin{aligned} J_k{}^\mu &= \frac{1}{2}\iint \langle p''|e^{i(\mathbf{k},\mathbf{x})}|p'\rangle\langle p'|\alpha^\mu\Theta(-E)|p''\rangle\, d^3p'\, d^3p'' \\ &= \frac{1}{2}\int \langle p'|\alpha^\mu\Theta(-E)|p' + k\rangle\, d^3p'. \end{aligned} \tag{20.4}$$

Now let us take first of all the case of no external field. For $\mathcal{A}_\mu = 0$, E and $\Theta(-E)$ commute with the three p's, so that $J_k{}^\mu$ has only diagonal elements when referred to the momentum representation: thus $J_k{}^\mu = 0$ for $k \neq 0$. For $k = 0$ we are not interested in the value of $J_k{}^\mu$, because our whole perturbation theory is concerned only with the values of k that are not zero. The zero value of k would correspond to static fields, which are not included in the perturbation.

Let us now consider the case of a uniform magnetic field. For a uniform magnetic field, the $\mathcal{A}_r$'s are linear in the x's. Then $\langle p'|E|p''\rangle = 0$ for p'' differing from p'. Under these conditions $|E|$ also involves no matrix elements differing appreciably from the diagonal, but only elements infinitely close to the diagonal, and we again have $J_k{}^\mu = 0$ for k differing from 0. Thus the Z term contributes nothing to the anomalous magnetic moment.

For the Lamb-shift calculation we have an electrostatic field that is not uniform, and in that case we do have off-diagonal elements contributing to $J_k{}^\mu$.

The Z terms are called the vacuum-polarization terms by the people who work in terms of the Schrödinger picture and Feynman diagrams. I do not know whether that name has much significance when one works in the Heisenberg picture, but still it is good notation. The vacuum-polarization term contributes nothing to the energy of the electron in the absence of a static field and nothing in a uniform magnetic field, but it does contribute something in a nonuniform electric field and thus gives a contribution to the Lamb shift. We shall have to work it out later when we come to work out the Lamb shift in detail.

Well, let us now study Y. We can simplify Y a little using the properties (19.1) of the Θ function, and then it becomes

$$Y_{n'i} = (4\pi)^{-1} \sum_n \int \langle n'|\alpha_\mu e^{i(\mathbf{k},\mathbf{x})}|n\rangle\langle n|\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})}|i\rangle \frac{\nu}{\nu|\mathbf{k}| + E_n - E_i} \times \left(\nu + \frac{E_n}{|E_n|}\right)\frac{d^3k}{|\mathbf{k}|}. \tag{20.5}$$

We can write this in a more concise form,

$$Y_{ni} = (4\pi)^{-1} \int \langle n|\alpha^\mu e^{i(\mathbf{k},\mathbf{x})} \frac{\nu + (E/|E|)}{|\mathbf{k}| + \nu E - \nu E_i} e^{-i(\mathbf{k},\mathbf{x})} \alpha_\mu|i\rangle \frac{d^3k}{|\mathbf{k}|}.$$

We can make a further simplification if we replace νE by $|E|$ in the denominator, which is permissible because the numerator vanishes when $\nu E = -|E|$. Y_{ni} can be looked upon as the n, i element of some operator

$$Y = (4\pi)^{-1} \int \alpha_\mu e^{i(\mathbf{k},\mathbf{x})} \frac{\nu + (E/|E|)}{|\mathbf{k}| + |E| - \nu E_i} e^{-i(\mathbf{k},\mathbf{x})} \alpha^\mu \frac{d^3k}{|\mathbf{k}|}. \tag{20.6}$$

For any given electron operator μ, we define $\tilde{\mu} = e^{-i(\mathbf{k},\mathbf{x})}\mu e^{i(\mathbf{k},\mathbf{x})}$. Then

$$Y = (4\pi)^{-1} \int \alpha_\mu \frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\mathbf{k}| + |\tilde{E}| - \nu E_i} \alpha^\mu \frac{d^3k}{|\mathbf{k}|}. \tag{20.7}$$

There is a change in sign in k in (20.7) compared with (20.6), but that does not matter because k is just a dummy variable of integration. The sign in (20.7) is the more convenient one to take.

§21. *Cutoffs*

The expression for Y involves an integration over k, and when you proceed to carry out that integration you get an infinity coming in. This working with the Heisenberg picture has eliminated many of the infinities but not all of them. There are still some which survive, and, of course, if we want to have a logical theory, we cannot tolerate any infinities at all. What we have to do now is to make a cutoff in the interaction energy so as to get rid of this infinity.

We are changing our Hamiltonian, of course, when we put this cutoff into the interaction energy. Now you may object that changing the Hamiltonian in this way spoils the relativistic invariance of the theory. That is quite true; however, if we do the cutoff properly the relativistic invariance will be spoiled only for high-energy phenomena.

Quantum electrodynamics in any case cannot be a completely accurate physical theory because it neglects all sorts of particles that turn up in high-energy processes, mesons, neutrinos, and so on. Quantum electrodynamics cannot be a complete theory in itself. We find that we cannot make quantum electrodynamics consistently relativistically invariant for high-energy processes. Well, that does not really matter. We shall find that we can make the cutoff at a fairly high energy value, of the order of several hundred million volts. It is reasonable to make the cutoff at an energy value somewhere there and we shall then have a theory that fails for processes involving energies of more than some hundreds of millions of volts, but which we can hope will be accurate for lower energies. We may thus hope that it will be accurate for calculating things like the anomalous magnetic moment and the Lamb shift, where there are no high-energy processes involved.

I look upon it in this way: We have a quantum electrodynamics that cannot be accurate for high-energy processes, because we do not know how to take into account high-energy interactions, and our cutting off the interaction for high energies is just an expression of our ignorance of how high-energy processes should be taken into account.

We have to be careful in how we cut off the interaction energy. The usual treatment of quantum field theory is quite sloppy over this point. I want to do it in an accurate logical way, and for that purpose we have to specify accurately just how the cutoff is to be made. When one looks into it, one sees that there is more than one cutoff process worthy of consideration.

Our interaction energy is $H_Q = -e\int \bar{\psi}\alpha_\mu \psi A^\mu \, d^3x$. We express it in terms of the Fourier coefficients and we see that each term involves three particles. It involves $\bar{\psi}_n$, ψ_n, $A_{\mu k}^{(\nu)}$, which are emission or absorption operators for three particles.

Now one way to cut off is just to put an upper limit to the magnitude of $|\mathbf{k}|$ in the interaction energy. I call it cutoff (a).

Cutoff (a): Limit $|\mathbf{k}|$, say $|\mathbf{k}| < g$, where g is some large number. That is the most natural and obvious way to apply a cutoff, and of course if you do that for an integral like Y you immediately get a finite quantity, and are no longer bothered by infinities. However, it turns out that this one is not so satisfactory, and one should consider other cutoff processes.

Let us take cutoff (b). Here we limit the sum of the moduli of the energies of the three particles involved in each term of H_Q; that means $|E_n| + |E_i| + |\mathbf{k}| < 2g$. We shall be interested in the final state being equal to or rather close to i; and the three energies involved are thus E_n, the energy of the intermediate state; E_i; and $|\mathbf{k}|$. E_i is a constant as far as the integration process is concerned, and so let us take it over to the other side and get $|E_n| + |\mathbf{k}| < 2g - E_i$. That is a reasonable method of applying the cutoff.

One should be careful, of course, when one thinks of various possible ways of applying the cutoff, that one preserves the reality of H_Q. You must not cut off H_Q in some way which would turn it into something that is not physically real.

There is a third cutoff process to be considered. This is where we limit the sum of the kinetic energies of the three particles. Previously we limited the total energies. I am now going to limit only the kinetic energies to get cutoff (c). E contained a part $-e\mathcal{A}_0$ which was potential energy and which we are now going to exclude in the cutoff process. That has the effect $|E + e\mathcal{A}_0| + |\mathbf{k}| + E_i + e\mathcal{A}_0 < 2g$.

There are thus various cutoff processes we might consider; one cannot say offhand which is the right one to use. We have to investigate each of them. We would like to have our cutoff process lead to physical results that are independent of just where we chose g. So we must carry on with the calculations and see what luck we are going to have in that respect, and we also have to decide later which of those three cutoff processes we are going to adopt. But already at this stage we can work out the differences these cutoff processes will lead to. I shall do that now and work out the differences in the expression Y according to these three processes.

Let us go back to the integral (20.5). Cutoff (b) involves the restricting of variables by the inequality $|E_n| + E_i + |\mathbf{k}| < 2g$. If we want to bring in such an inequality, the easiest way is to introduce into the integrand in (20.5) the factor $\Theta(2g - |E_n| - E_i - |\mathbf{k}|)$. Equation (20.6) then becomes

$$Y_{(b)} = (4\pi)^{-1} \int \alpha_\mu e^{i(\mathbf{k},\mathbf{x})} \frac{\nu + (E/|E|)}{|\mathbf{k}| + |E| - \nu E_i} \times \Theta(2g - |E| - E_i - |\mathbf{k}|)\alpha^\mu e^{-i(\mathbf{k},\mathbf{x})} \frac{d^3k}{|\mathbf{k}|}, \tag{21.1}$$

and (20.7) becomes

$$Y_{(a)} = (4\pi)^{-1} \int \alpha_\mu \frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\mathbf{k}| + |\tilde{E}| - \nu E_i}$$
$$\times \; \Theta(2g - |\tilde{E}| - E_i - |\mathbf{k}|)\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \cdot$$

Similarly for the cutoff (a) we get

$$Y_{(a)} = (4\pi)^{-1} \int \alpha_\mu \frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\mathbf{k}| + |\tilde{E}| - \nu E_i} \Theta(g - |\mathbf{k}|)\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \cdot$$

We have $E = \alpha_m m + (\boldsymbol{\alpha}, \boldsymbol{\pi}) - e\mathcal{A}_0$, where $\pi_r = p_r + e\mathcal{A}_r$. Thus

$$\tilde{E} = \alpha_m m + (\boldsymbol{\alpha}, \boldsymbol{\pi}) + (\boldsymbol{\alpha}, \mathbf{k}) - e\mathcal{A}_0,$$
$$\tilde{E}^2 = k^2 + 2(\mathbf{k}, \boldsymbol{\pi}) - 2(\boldsymbol{\alpha}, \mathbf{k})e\mathcal{A}_0 + \text{terms independent of } k.$$

So for large $|k|$, of the order g,

$$|\tilde{E}| = |k| + (\mathbf{l}, \boldsymbol{\pi}) - (\boldsymbol{\alpha}, \mathbf{l})e\mathcal{A}_0 + O(g^{-1})$$

where $l_r = k_r/|k|$. Thus

$$\frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\mathbf{k}| + |\tilde{E}| - \nu E_i} = \frac{\nu + (\boldsymbol{\alpha}, \mathbf{l})}{2|k|} + O(g^{-2})$$
$$= \frac{(\boldsymbol{\alpha}, \mathbf{l})}{2|\mathbf{k}|} + O(g^{-2})$$

when one averages over ν. Also

$$\Theta(2g - |\tilde{E}| - E_i - |k|) = \Theta\big(2g - 2|k| - (\mathbf{l}, \boldsymbol{\pi})$$
$$+ (\boldsymbol{\alpha}, \mathbf{l})e\mathcal{A}_0 - E_i + O(g^{-1})\big)$$
$$= \Theta(g - |\mathbf{k}| + F_{(b)})$$

say, where

$$F_{(b)} = -\tfrac{1}{2}(\mathbf{l}, \boldsymbol{\pi}) + \tfrac{1}{2}(\boldsymbol{\alpha}, \mathbf{l})e\mathcal{A}_0 - \tfrac{1}{2}E_i + O(g^{-1}).$$

Hence

$$Y_{(b)} - Y_{(a)} = (4\pi)^{-1} \int \alpha_\mu \frac{(\boldsymbol{\alpha}, \mathbf{l})}{|2\mathbf{k}|}$$
$$\times \{\Theta(g - |\mathbf{k}| + F_{(b)}) - \Theta(g - |\mathbf{k}|)\}\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \cdot$$

Let us put $d^3k = |\mathbf{k}|^2 \, d|\mathbf{k}| \, d\Omega$, where $d\Omega$ = element of solid angle. Then we carry through the integration first with respect to $|k|$ and then over all solid angles. We get

$$Y_{(b)} - Y_{(a)} = (4\pi)^{-1}\alpha_\mu \int \tfrac{1}{2}(\boldsymbol{\alpha}, \mathbf{l})F_{(b)}\, d\Omega\, \alpha^\mu .$$

This quantity $F_{(b)}$ is, of course, an operator, but that doesn't invalidate the argument. For each value of the l's, that is to say, for each direction in k-space, we may take a representation for which $F_{(b)}$ is diagonal; then the integration over $|k|$ becomes trivial. If we use a different representation, the result must be the same.

When we carry out the integration over all solid angles, many of the terms will vanish. The only terms of $F_{(b)}$ that contribute anything are those which are linear in the l's. We get

$$Y_{(b)} - Y_{(a)} = \tfrac{1}{2}\alpha_\mu \left[-\frac{(\boldsymbol{\alpha}, \boldsymbol{\pi})}{6} + \tfrac{1}{2} e\mathcal{A}_0 \right] \alpha^\mu .$$

Then carrying out the summation over the four values of μ gives us the final result

$$Y_{(b)} - Y_{(a)} = -\frac{(\boldsymbol{\alpha}, \boldsymbol{\pi})}{6} - \tfrac{1}{2} e\mathcal{A}_0 . \tag{21.2}$$

With cutoff (c) we follow the same method; we just have to replace E by the kinetic part of E. Thus we replace E by $E + e\mathcal{A}_0$, and E_i by $|E_i + e\mathcal{A}_0|$ in the argument of Θ in (21.1). Then we have to replace $F_{(b)}$ by $F_{(c)}$, which is $F_{(c)} = -\tfrac{1}{2}(\mathbf{l}, \boldsymbol{\pi}) - \tfrac{1}{2}|E_i + e\mathcal{A}_0|$. That gives us the result

$$Y_{(c)} - Y_{(a)} = -\tfrac{1}{6}(\boldsymbol{\alpha}, \boldsymbol{\pi}). \tag{21.3}$$

Well that is the method of calculating the differences caused by the different methods of cutoff. You note that the differences don't depend on g. So long as g is large, these results are valid, and different kinds of cutoff lead to finite differences in our expression for Y. For the present we're going to leave open the question of which of these cutoffs is the correct one, and we shall decide that later.

§22. *Case of No Static Field*

Let us go on now to take the case of no static field and work out Y explicitly. With $\mathcal{A}_\mu = 0$, we have

$$E = (\boldsymbol{\alpha}, \mathbf{p}) + \alpha_m m, \qquad E^2 = p^2 + m^2,$$

and $|E| = (p^2 + m^2)^{1/2} = p_0, p_0$ being defined as the positive square root of $p^2 + m^2$. So (20.6) becomes

$$Y = (4\pi)^{-1} \int \alpha_\mu \frac{\nu\tilde{p}_0 + (\alpha, \tilde{\mathbf{p}}) + \alpha_m m}{\tilde{p}_0(|\mathbf{k}| + \tilde{p}_0 - \nu E_i)} \alpha^\mu \frac{d^3k}{|\mathbf{k}|}.$$

Let

$$Y_0 = (4\pi)^{-1} \int \alpha_\mu \frac{\nu\tilde{p}_0 + (\alpha, \tilde{\mathbf{p}}) + \alpha_m m}{\tilde{p}_0(|\mathbf{k}| + \tilde{p}_0 - \nu p_0)} \alpha^\mu \frac{d^3k}{|\mathbf{k}|}.$$

It differs from Y in having p_0 instead of E_i in the denominator. Now we need to know Y only to the extent that we need $Y|i\rangle$, which suffices to fix all the matrix elements Y_{ni}. We have $p_0|i\rangle = E_i|i\rangle$, so $Y_0|i\rangle = Y|i\rangle$, and thus Y_0 is as good as Y for giving us the results we want. We shall proceed to evaluate the integral Y_0 instead of Y. This way of setting out the work is convenient for the later applications.

We can carry out this summation over μ fairly easily. We have

$$\begin{aligned} \tfrac{1}{2}\alpha_\mu\alpha^\mu &= -1, \\ \tfrac{1}{2}\alpha_\mu\alpha_r\alpha^\mu &= \alpha_r, \qquad r = 1, 2, 3, \\ \tfrac{1}{2}\alpha_\mu\alpha_m\alpha^\mu &= 2\alpha_m. \end{aligned} \tag{22.1}$$

This gives us

$$Y_0 = (2\pi)^{-1} \int \frac{-\nu\tilde{p}_0 + (\alpha, \tilde{\mathbf{p}}) + 2\alpha_m m}{\tilde{p}_0(|\mathbf{k}| + \tilde{p}_0 - \nu p_0)} \frac{d^3k}{|\mathbf{k}|}.$$

And this we can write in a more convenient form

$$Y_0 = (2\pi)^{-1} \int \frac{\nu|\mathbf{k}| - p_0 + (\alpha, \tilde{\mathbf{p}}) + 2\alpha_m m}{\tilde{p}_0(|\mathbf{k}| + \tilde{p}_0 - \nu p_0)} \frac{d^3k}{|\mathbf{k}|},$$

which equals the previous one because the difference between them vanishes when one averages over ν.

To evaluate this integral we can proceed as follows. We choose a system of coordinates for which p_1 and p_2 are zero; i.e., we choose our k-axis to be the direction p_3. Now you might say: Is such a thing justifiable when one takes into account that the p's are not just numbers but are operators? In general that is not justifiable; but in this piece of work the p's commute with everything, so we could very well work in the representation in which the p's are diagonal and then take p_1 and p_2 to be zero. Later on, when we have to do corresponding calculations where we have field functions coming in which don't commute with the p's, the situation is more complicated and we cannot use this trick.

We then have

$$Y_0 = (2\pi)^{-1}\int \frac{\nu|\mathbf{k}| - p_0 + \alpha_3(p_3 + k_3) + 2\alpha_m m}{\tilde{p}_0(|\mathbf{k}| + \tilde{p}_0 - \nu p_0)}\frac{d^3k}{|\mathbf{k}|}.$$

We neglect terms linear in k_1, k_2, because upon integration over all k-values, terms linear in k_1, k_2 will contribute nothing, because of the symmetry between positive and negative k-values. Of course, we cannot neglect terms that are linear in k_3, because $\tilde{p}_0$ involves k_3:

$$\begin{aligned}\tilde{p}_0 &= [m^2 + (p_3 + k_3)^2 + k_1{}^2 + k_2{}^2]^{1/2}\\ &= (m^2 + p_3{}^2 + 2p_3k_3 + |\mathbf{k}|^2)^{1/2}.\end{aligned}$$

Now one can carry out the integration by making a suitable change of variables. The trick is to introduce a variable $\omega = \tilde{p}_0 + |\mathbf{k}|$. First we pass from the three k's to cylindrical variables. We put

$$k_1 = \rho\cos\phi, \qquad k_2 = \rho\sin\phi, \qquad \rho = (k_1{}^2 + k_2{}^2)^{1/2},$$

and we pass from k_1, k_2, k_3 to k_3, ρ, ϕ. We have $d^3k = \rho\, d\rho\, d\phi\, dk_3$. Then we pass to ω, ϕ, k_3. Now for the second passage here we need a Jacobian,

$$\frac{\partial(\omega k_3)}{\partial(\rho k_3)} = \left(\frac{\partial\omega}{\partial\rho}\right)_{k_3=\text{const}} = \frac{\rho}{\tilde{p}_0} + \frac{\rho}{|\mathbf{k}|} = \frac{\rho\omega}{\tilde{p}_0|\mathbf{k}|}.$$

And we then get $d^3k/|\mathbf{k}| = (\tilde{p}_0/\omega)\, d\omega\, d\phi\, dk_3$.

Well, carrying out that change of variables, the expression for Y_0 becomes

$$Y_0 = \iint \frac{\alpha_3 p_3 + 2\alpha_m m - p_0 + \nu|\mathbf{k}| + \alpha_3 k_3}{\omega - \nu p_0}\frac{d\omega}{\omega}\, dk_3. \qquad (22.2)$$

Let us now work with the cutoff (b), because it is the convenient one to use here. Then we have to integrate first with respect to k_3 and finally with respect to ω. You see, ω is just the total energy variable apart from the constant term E_i, so the cutoff with respect to total energy means a cutoff with respect to the ω variable.

Now it is fairly straightforward integration. First, to integrate with respect to k_3, we must know what the limits of k_3 are for a given ω. For a given ω, the limits of k_3 occur when $k_3 = \pm|\mathbf{k}|$. We have this connection: $(\omega - |k|)^2 = p_0{}^2 + 2p_3k_3 + |\mathbf{k}|^2$; that gives $2\omega|\mathbf{k}| = \omega^2 - p_0{}^2 - 2p_3k_3$. Then you see that the limits of integration are

$k_3 = (\omega^2 - p_0{}^2)/2(\omega + p_3)$ and $-(\omega^2 - p_0{}^2)/2(\omega - p_3)$. Now if you look at the expression (22.2) for Y_0, you see that there are some terms independent of k_3, some linear in k_3, and some linear in $|\mathbf{k}|$; and so we need three integrals,

$$\int dk_3 = \frac{\omega^2 - p_0{}^2}{\omega^2 - p_3{}^2}\,\omega,$$

$$\int k_3\, dk_3 = -\frac{1}{2}\frac{(\omega^2 - p_0{}^2)^2}{(\omega^2 - p_3{}^2)^2}\,\omega p_3, \qquad \int |\mathbf{k}|\, dk_3 = \frac{(\omega^2 - p_0{}^2)^2}{2(\omega^2 - p_3{}^2)^2}\,\omega^2.$$

This gives us

$$\left\{Y_0 = \int \frac{\alpha_3 p_3 + 2\alpha_m m - p_0}{\omega^2 - p_3{}^2} + \nu\frac{(\omega^2 - p_0{}^2)\omega}{2(\omega^2 - p_3{}^2)^2} - \frac{\alpha_3 p_3(\omega^2 - p_0{}^2)}{2(\omega^2 - p_3{}^2)^2}\right\}$$
$$\times\, (\omega + \nu p_0)\, d\omega.$$

At this stage it is convenient to do the ν averaging. We could have done this ν-averaging at any time during the calculation, but it would be inconvenient to do it earlier; it would mean our terms would get longer. But here we can do it conveniently. We find

$$Y_0 = \int\!\!\int \left[\frac{\alpha_3 p_3 + 2\alpha_m m - p_0}{\omega^2 - p_3{}^2} + \frac{(p_0 - \alpha_3 p_3)(\omega^2 - p_0{}^2)}{2(\omega^2 - p_3{}^2)^2}\right] \omega\, d\omega.$$

Now the $\omega^2 - p_0{}^2$ inside the [ ] can be replaced by $\omega^2 - p_3{}^2 - m^2$, and we get

$$Y_0 = \frac{1}{2}\int\!\!\int \left[\frac{\alpha_3 p_3 + 4\alpha_m m - p_0}{\omega^2 - p_3{}^2} - \frac{(p_0 - \alpha_3 p_3)m^2}{(\omega^2 - p_3{}^2)^2}\right] \omega\, d\omega.$$

This is something we can quite easily integrate with respect to ω and the result is, when we put in the limits of integration,

$$Y_0 = \frac{1}{4}\left[(\alpha_3 p_3 + 4\alpha_m m - p_0)\log(\omega^2 - p_3{}^2) + \frac{p_0 - \alpha_3 p_3}{\omega^2 - p_3{}^2}\, m^2\right]_{p_0}^{2g-E}$$
$$= \tfrac{1}{2}(\alpha_3 p_3 + 4\alpha_m m - p_0)\log(2g/m) - \tfrac{1}{4}(p_0 - \alpha_3 p_3).$$

We did this calculation for the case of p_1 and p_2 vanishing, but we can go back to general p's. For general p_1, p_2, p_3 obviously we just have to replace $\alpha_3 p_3$ by the scalar product of $(\boldsymbol{\alpha}, \mathbf{p})$. We get the general expression

$$Y_0 = \tfrac{1}{2}[(\boldsymbol{\alpha}, \mathbf{p}) + 4\alpha_m m - p_0]\log(2g/m) - \tfrac{1}{4}[p_0 - (\boldsymbol{\alpha}, \mathbf{p})]. \qquad (22.3)$$

Now we have $[(\boldsymbol{\alpha}, \mathbf{p}) + \alpha_m]|i\rangle = E_i|i\rangle = p_0|i\rangle$. We then find that

$$Y|i\rangle = \alpha_m m[\tfrac{3}{2}\log(2g/m) - \tfrac{1}{4}]|i\rangle = qm\alpha_m|i\rangle,$$

where $q = \frac{3}{2}\log(2g/m) - \frac{1}{4}$.

That is the result of this calculation. Let's see what effect it has. From (19.2),

$$i\frac{\partial K_2{}^*}{\partial t} = \frac{e^2}{\pi} qm \sum_n \bar{\psi}_n\langle n|\alpha_m|i\rangle e^{i(E_n - E_i)t}.$$

This shows us the way in which the perturbation affects the emission operator we're interested in.

§23. *Mass Renormalization*

This effect can be eliminated by a procedure known as *mass renormalization.* This procedure consists of adding a further term to the Hamiltonian corresponding to a variation in the mass parameter. Let us keep m to be the observed mass. In H replace m by $m + \delta m$; that is, the mass parameter as it occurs in the Hamiltonian is now going to be different from the observed mass. This variation in the Hamiltonian is going to be treated as a second-order perturbation. We get an extra term in H of amount

$$H_2 = \delta m \int \bar{\psi}_x \alpha_m \psi_x \, d^3x = \delta m \sum_{nn'} \bar{\psi}_n \langle n|\alpha_m|n'\rangle \psi_{n'}.$$

This is to be treated as an extra second-order perturbation, giving an extra variation in $K_2{}^*$,

$$\left(i\frac{\partial K_2{}^*}{\partial t}\right)_{\text{extra}} = [H_2{}^*, \bar{\psi}_i]_-,$$

where

$$H_2{}^* = \delta m \sum_{nn'} \bar{\psi}_n \langle n|\alpha_m|n'\rangle \psi_{n'} e^{i(E_n - E_{n'})t}.$$

Using this $H_2{}^*$ in the commutator, we get

$$\left(i\frac{\partial K_2{}^*}{\partial t}\right)_{\text{extra}} = \delta m \sum_n \bar{\psi}_n \langle n|\alpha_m|i\rangle e^{i(E_n - E_i)t}.$$

This extra $i(\partial K_2{}^*/\partial t)$ is to be added to the previous one.

Now we can make the total $i(\partial K_2{}^*/\partial t) = 0$, provided we choose δm suitably, that is, provided we take $\delta m = (-e^2/\pi)qm$. This procedure is mass renormalization. The result is that the perturbation does not effect the electron emission operator to the second order of accuracy.

At this stage we might discuss what value we should take for g; g must not be too large because we have to be able to treat $H_2{}^*$ as a perturbation; that stops us from making g go to infinity. But we can take g to be fairly large and still keep $\delta m/m$ rather small, because of the logarithmic factor. In our units, we have $e^2 = 1/137$. I worked out a few figures. With $2g/m = 1000$ we get $\log (2g/m) = 6.9$ and $\delta m/m = 1/43$, which is small enough to be reasonably counted as a perturbation effect. We might take $2g/m = 10.000$; then $\log (2g/m) = 9.2$ and $\delta m/m = 1/32$, which is still reasonably small for a perturbation theory. g is of the order of $250 \times 10^6\, eV$ in the first case and 10 times as big in the second. One must expect electrodynamics to break down with energies somewhere between these two orders because there will be further effects coming in at these high energies due to the possibility of creating mesons, neutrinos, etc. So we should imagine $2g/m$ to have a value somewhere in this range, which means that our electrodynamics is limited from the point of view of high-energy processes. But we may expect it to be a pretty good theory for processes with energies small compared with these energies.

With this cutoff, $\delta m/m$ can reasonably be counted as a perturbation effect. That I think is how the mass renormalization ought to be handled. Handled in this way we do get a logical theory where we're solving equations, not merely using working rules, and we're neglecting only terms we can feel are small and for which the neglect is justifiable.

Now this calculation is done with a cutoff (b). How will things go if we use a different cutoff? You see that we did use cutoff (b), because our last variable of integration was ω, equal to $|\mathbf{k}| + \tilde{p}_0$, and the cutoff of this variable means a limitation of $|\mathbf{k}| + \tilde{p}_0$. That just corresponds to what we have with the cutoff (b).

With cutoff (a) we get, according to (21.2), $Y = q\alpha_m m + \frac{1}{6}(\boldsymbol{\alpha}, \mathbf{p})$. Now we can get rid of the first term by mass renormalization but we can't do anything about $(\boldsymbol{\alpha}, \mathbf{p})$. You might say: Can we not introduce a further term in the Hamiltonian, a sort of counterterm to $(\boldsymbol{\alpha}, \mathbf{p})$ which will cut it out? We could introduce such a further term, but that would have to be of the form of some multiple of $(\boldsymbol{\alpha}, \mathbf{p})$, and such a further term is not of a relativistic nature. If we put in such a counterterm in the Hamiltonian to get rid of $(\boldsymbol{\alpha}, \mathbf{p})$ our theory would no longer be relativistic even for small-energy processes. Well, we infer that cutoff (a) is a bad cutoff.

At present we cannot distinguish between cutoff (b) and cutoff (c), because they are equivalent for the case of no static field. But we can distinguish between them by an argument depending on gauge invariance. Let us suppose the static magnetic potential is zero, $\mathcal{A}_r = 0$, and the static electric potential is constant in space as well as in time, $\mathcal{A}_0 =$ a c-number. Now such a potential, of course, means no field at all. It ought not to affect things at all. Let us bring such a potential into our equations and see what effect it has on them. This means working with an energy operator

$$E = (\boldsymbol{\alpha}, \mathbf{p}) + \alpha_m m - e\mathcal{A}_0 = U - e\mathcal{A}_0,$$

where U is the energy operator in the case of no potentials.

Now in evaluating (20.6) we are working with the quantity

$$\frac{\nu + (E/|E|)}{|\mathbf{k}| + |E| - \nu E_i}, \tag{23.1}$$

multiplying it by suitable factors on the left and the right, and then integrating; and we must see how the result is affected by bringing in the external potential. Let us make the restriction $|e\mathcal{A}_0| < m$ for simplicity. This means that the potential must not be a terribly big thing; but it still can be fairly big. Any eigenfunction of U is an eigenfunction of E, because the difference between U and E is just a c-number. If $U > 0$, then $E > 0$, in virtue of the inequality $|e\mathcal{A}_0| < m$; therefore, $E/|E| = U/|U| = U/p_0$, where p_0, you remember, is just $(p^2 + m^2)^{1/2}$.

First, let us replace $|E|$ by νE in the denominator of (23.1); we get $[\nu + (E/|E|)]/[|\mathbf{k}| + \nu(E - E_i)]$. The replacement is permissible, because $|E| = \pm\nu E$ and if $|E| = -\nu E$ we have the numerator vanishing, so that the equation is valid in any case. Let us now write it $[\nu + (U/p_0)]/[|\mathbf{k}| + \nu(U - e\mathcal{A}_0 - E_i)]$. Now we can make the same sort of change again, replacing νU by $|U| = p_0$, using the same argument again. We thus get $[\nu + (U/p_0)]/[|\mathbf{k}| + p_0 - \nu(e\mathcal{A}_0 + E_i)]$.

The result we have here is the same as in the case where there is no static potential, except that the number E_i is replaced by $E_i + e\mathcal{A}_0$. In the evaluation of the integral you see that everything would go as before, with a number $E_i + e\mathcal{A}_0$ replacing the previous number E_i. So our previous calculations for the case of no potentials can be taken over with this substitution.

When we previously carried out the calculation, we made use of the weak equation $(\alpha, \mathbf{p}) + \alpha_m m \approx E_i$, valid with both sides applied to $|i\rangle$. Now we have instead

$$(\alpha, \mathbf{p}) + \alpha_m m \approx E_i + e\mathcal{A}_0,$$

because we still have $E \approx E_i$. So here again we have to substitute the number $E_i + e\mathcal{A}_0$ for E_i. The whole calculation goes through as before with this substitution, and we get the final result that $Y \approx q\alpha_m m$, with the same q as before. With mass renormalization we get $dK_2{}^*/dt = 0$, as before.

You see everything goes through all right with the substitution. We ought to see what kind of limiting process we are now using. Previously we were using a variable of integration defined as $\omega = \tilde{p}_0 + |\mathbf{k}|$, and we did the integration with respect to ω last. That means we had a cutoff with respect to ω. Now this variable ω is unaffected by the substitution of $E_i + e\mathcal{A}_0$ for E_i, so we are still working with a cutoff with respect to ω. The variable ω just refers to the total kinetic energy, so we are working with cutoff (c).

Cutoff (c) is the physically correct cutoff which gives a gauge-invariant result. Cutoff (b) is not gauge-invariant.

We did this calculation using an electrostatic potential constant in space and time. You might say: What would happen if we used a magnetic potential that is constant in space and time? You easily see that that has quite a trivial effect on the calculation; it just means replacing p_r by π_r. After the integration p_r disappeared from the final result, so the π_r will disappear equally well. So a constant magnetic potential has no influence on the calculation. Well, that completes the theory for the case of no static field or the case of constant potentials.

§24. *The Anomalous Magnetic Moment*

We now go on to consider cases in which these potentials are not constant. Let us first consider the problem of the anomalous magnetic moment. For that purpose we have to take the potentials $\mathcal{A}_r$ to represent a uniform magnetic field, uniform and constant over the whole of space, and take $\mathcal{A}_0 = 0$.

Let us suppose the magnetic field is of magnitude $\mathcal{H}$ in the direction of x_3. Now we have

$$E = (\alpha, \mathbf{p} + e\mathcal{A}) + \alpha_m m = (\alpha, \pi) + \alpha_m m,$$

where π, as before, is $p + e\mathcal{A}$. From the one-particle theory we get $E^2 = \pi^2 + m^2 + e\mathcal{H}\sigma_3$, where $\alpha_1\alpha_2 = i\sigma_3$.

We shall assume $\mathcal{H}$ is small and neglect $\mathcal{H}^2$. Thus

$$|E| = (m^2 + \pi^2)^{1/2} + \tfrac{1}{2}e\mathcal{H}\sigma_3(m^2 + \pi^2)^{-1/2}.$$

We have

$$Y = (4\pi)^{-1} \int \alpha_\mu \mathcal{Y} \alpha^\mu \frac{d^3k}{|\mathbf{k}|},$$

with

$$\mathcal{Y} = \left(\nu + \frac{\tilde{E}}{|\tilde{E}|}\right)(|\tilde{E}| + |\mathbf{k}| - \nu E_i)^{-1}. \tag{24.1}$$

We shall suppose the momentum of the electron is small, so the π's are small, and neglect terms of the third degree in the π's. We shall also neglect $\pi\mathcal{H}$.

The denominator in $\mathcal{Y}$ sometimes vanishes if $\nu = 1$. It cannot vanish for $\nu = -1$. We must work out the order of magnitude of the contribution to Y of the domain of integration for which the denominator is small. We shall find that it is negligible.

We have

$$\begin{aligned}(|\tilde{E}| + |\mathbf{k}| - E_i)^{-1} \\ &= (|\tilde{E}| - |\mathbf{k}| + E_i)(\tilde{E}^2 - k^2 - E_i^2 + 2|\mathbf{k}|E_i)^{-1} \\ &= (|\tilde{E}| - |\mathbf{k}| + E_i)[E^2 - E_i^2 + 2(k\pi) + 2|\mathbf{k}|E_i]^{-1}.\end{aligned}$$

Write

$$D = \tfrac{1}{2}(E^2 - E_i^2) + (k\pi) + |\mathbf{k}|E_i.$$

We can work with this denominator instead of the one in (24.1) with $\nu = 1$.

We need to know Y only to the extent that we need $Y|i\rangle$. We shall write

$$A \approx B \quad \text{if} \quad A|i\rangle = B|i\rangle.$$

Let α_λ denote a longitudinal α with respect to the direction of the magnetic field, $\alpha_\lambda = 1$ or α_3. Let α_τ denote a transverse α with respect to the direction of the field, $\alpha_\tau = \alpha_1$ or α_2. Then α_λ commutes with E^2,

$$\alpha_\lambda E^2 \alpha_\lambda = E^2,$$

but

$$\alpha_\tau E^2 \alpha_\tau = \pi^2 + m^2 - e\mathcal{H}\sigma_3 = E^2 - 2e\mathcal{H}\sigma_3.$$

In these equations there is no summation for λ or τ.

We have $\alpha_\lambda D \alpha_\lambda \approx (k\pi) + |\mathbf{k}|E_i$. Thus $\alpha_\lambda D \alpha_\lambda$ cannot vanish, because of the smallness of $(k\pi)$. We have

$$\alpha_\tau D \alpha_\tau \approx (k\pi) + |\mathbf{k}|E_i - e\mathcal{H}\sigma_3, \tag{24.2}$$

and this can vanish for small values of $|\mathbf{k}|$ of order $\mathcal{H}$. Its vanishing corresponds physically to the electron emitting or absorbing a photon and changing its spin direction.

For the transverse α_τ, the numerator in $\alpha_\tau \tilde{\mathcal{Y}} \alpha_\tau$ (with $\nu = 1$) is small. It is (apart from a factor of order unity)

$$\alpha_\tau\{1 + [(\alpha\pi) + (\alpha k) + \alpha_m m]|\tilde{E}|^{-1}\}\alpha_\tau,$$

which is of order $(1 - \alpha_m)$ + terms of order π or k, and $(1 - \alpha_m)$ applied to $|i\rangle$ is also of order π.

The domain of integration for which the denominator (24.2) is small can be pictured as a thick eggshell enclosing the origin, with a radius of order $e\mathcal{H}/m$. We get the order of magnitude of the integral over this domain if we replace the thick eggshell by a thin one, putting

$$\delta[(k\pi) + |\mathbf{k}|E_i - e\mathcal{H}\sigma_3] \qquad \text{for } [(k\pi) + |\mathbf{k}|E_i - e\mathcal{H}\sigma_3]^{-1}.$$

We then get for the integral

$$\int [O(k) + O(\pi)]\,\delta[(k\pi) + |\mathbf{k}|E_i - e\mathcal{H}\sigma_3]\frac{d^3k}{|\mathbf{k}|} = O(k^2) + O(k\pi).$$

Here we must substitute for k its value in the shell, which is of order $\mathcal{H}$. The result is negligible.

Define

$$\begin{aligned}\tilde{\mathcal{Y}}_0 = {} & \{\nu + [(\alpha\tilde{\pi}) + \alpha_m m] \\ & \times (m^2 + \tilde{\pi}^2)^{-1/2}\}[(m^2 + \tilde{\pi}^2)^{1/2} + |\mathbf{k}| - \nu(m^2 + \pi^2)^{1/2}]^{-1},\end{aligned}$$

with $\tilde{\pi}_r = \pi_r + k_r$, and

$$Y_0 = (4\pi)^{-1}\int \alpha_\mu \tilde{\mathcal{Y}}_0 \alpha^\mu \frac{d^3k}{|\mathbf{k}|}.$$

Suppose the π's are small and expand $\tilde{\mathfrak{Y}}_0$ and Y_0 as power series in the π's as far as terms of the second degree. The only second-degree terms in $\tilde{\mathfrak{Y}}_0$ involve $(\alpha\pi)(k\pi)$ or $(k\pi)^2$ or π^2. On integrating over all k-directions $(\alpha\pi)(k\pi)$ gives zero and $(k\pi)^2$ gives $\frac{1}{3}k^2\pi^2$. So the terms of second degree in Y_0 all involve π^2. Hence to this accuracy Y_0 is the same function of the π's whether the π's commute or not.

If we look upon the π's as commuting, $\tilde{\mathfrak{Y}}_0$ is the same as the corresponding expression of Section 22 with π_r for p_r, and so Y_0 is given by the Y_0 of (22.3) with π_r for p_r which gives

$$\begin{aligned} Y_0 = {} & \tfrac{1}{2}[(\alpha\pi) + \alpha_m m - m - (\pi^2/2m)][\log(2g/m) + \tfrac{1}{2}] \\ & + \alpha_m m[\tfrac{3}{2}\log(2g/m) - \tfrac{1}{4}]. \end{aligned} \tag{24.3}$$

This is obtained with the cutoff (b) or (c), because the last variable of integration is $\omega = |\mathbf{k}| + (m^2 + \tilde{\pi}^2)^{1/2}$, which differs by a negligible amount from $|\mathbf{k}| + |\tilde{E}|$.

Let us write $e\mathcal{H}\sigma_3 = h$ for brevity. We have from (24.1),

$$\begin{aligned} \tilde{\mathfrak{Y}} = {} & \{\nu + [(\alpha\tilde{\pi}) + \alpha_m m](m^2 + \tilde{\pi}^2 + h)^{-1/2}\} \\ & \times [(m^2 + \tilde{\pi}^2 + h)^{1/2} + |\mathbf{k}| - \nu E_i]^{-1}. \end{aligned}$$

$\tilde{\mathfrak{Y}}_0$ is the same expression with $h = 0$ and $(m^2 + \pi^2)^{1/2}$ for E_i.

We shall expand the difference $\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0$ in terms of the small quantities h and $E_i - (m^2 + \pi^2)^{1/2}$. The latter is weakly small, since

$$E_i - (m^2 + \pi^2)^{1/2} \approx |E| - (m^2 + \pi^2)^{1/2} = \tfrac{1}{2}h(m^2 + \pi^2)^{-1/2}. \tag{24.4}$$

In the expansion we do not have $E_i - (m^2 + \pi^2)^{1/2}$ occurring on the extreme right, so we cannot immediately use the expression (24.4) for it. We get, ignoring some factors on the left,

$$[E_i - (m^2 + \pi^2)^{1/2}][(m^2 + \tilde{\pi}^2)^{1/2} + |\mathbf{k}| - \nu(m^2 + \pi^2)^{1/2}]^{-1}.$$

This differs from the expression

$$[(m^2 + \tilde{\pi}^2)^{1/2} + |\mathbf{k}| - \nu(m^2 + \pi^2)^{1/2}]^{-1}[E_i - (m^2 + \pi^2)^{1/2}],$$

to which (24.4) is applicable, by quantities involving $[(\pi k), \pi^2]$, and such quantities are of order $\pi\mathcal{H}$ and are negligible. Thus the expansion is justified.

In the expansion of $\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0$ with the use of the weak equation (24.4), every term involves h or $\mathcal{H}$. Thus we can neglect the π's altogether, because we are neglecting $\pi\mathcal{H}$.

Put

$$(m^2 + k^2)^{1/2} = \mu.$$

Then

$$\begin{aligned}\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0 = {} & -\alpha_m m(h/2\mu^3)(\mu + |\mathbf{k}| - \nu m)^{-1} - (\nu + \alpha_m m/\mu) \\ & \times (\mu + |\mathbf{k}| - \nu m)^{-2}[(h/2\mu) - \nu(E_i - m)].\end{aligned}$$

We have

$$\begin{aligned}\alpha_\mu \alpha^\mu &= -2, & \alpha_\mu \alpha_m \alpha^\mu &= 4\alpha_m, \\ \alpha_\mu \sigma_3 \alpha^\mu &= 2\sigma_3, & \alpha_\mu \alpha_m \sigma_3 \alpha^\mu &= 0.\end{aligned}$$

Thus

$$\begin{aligned}\alpha_\mu(\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0)\alpha^\mu &= -(\mu + |\mathbf{k}| - \nu m)^{-2}[\nu(h/\mu) + 2(E_i - m) \\ &\qquad -4\nu\alpha_m(m/\mu)(E_i - m)] \\ &\approx -(\mu + |\mathbf{k}| - \nu m)^{-2}[\nu(h/\mu) + (h/m) - 2\nu(h/\mu)] \\ &= h(\mu + |\mathbf{k}| - \nu m)^{-2}(\nu m - \mu)/m\mu.\end{aligned}$$

Now

$$\begin{aligned}(\mu + |\mathbf{k}| - \nu m)(\mu - |\mathbf{k}| + \nu m) &= \mu^2 - k^2 - m^2 + 2\nu|\mathbf{k}|m \\ &= 2\nu|\mathbf{k}|m.\end{aligned}$$

Squaring,

$$(\mu + |\mathbf{k}| - \nu m)^2[\mu^2 + k^2 + m^2 + 2\nu m(\mu - |\mathbf{k}|) - 2\mu|\mathbf{k}|] = 4k^2m^2$$

or

$$(\mu + |\mathbf{k}| - \nu m)^2(\mu + \nu m)(\mu - |\mathbf{k}|) = 2k^2m^2.$$

Thus

$$\begin{aligned}\alpha_\mu(\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0)\alpha^\mu &\approx h(\mu + \nu m)(\mu - |k|)(\nu m - \mu)/2m^3k^2\mu \\ &= -h(\mu - |\mathbf{k}|)/2m^3\mu, \\ Y - Y_0 &= -(h/2m^3)(4\pi)^{-1}\int (1 - |\mathbf{k}|/\mu)\, d^3k/|\mathbf{k}| \\ &= -(h/2m^3)\int (1 - |\mathbf{k}|/\mu)|\mathbf{k}|\, d|\mathbf{k}|.\end{aligned}$$

It does not matter which cutoff we use for evaluating this small quantity. Use cutoff (a). Then we get for the integral,

$$\begin{aligned}\tfrac{1}{2}\{k^2 - k(m^2 + k^2)^{1/2} + m^2 \log [k + (m^2 + k^2)^{1/2}]\}_0^g \\ = \tfrac{1}{2}m^2[\log (2g/m) - \tfrac{1}{2}].\end{aligned}$$

Thus

$$Y - Y_0 = -(h/4m)[\log (2g/m) - \tfrac{1}{2}].$$

Taking the expression for Y_0 given by (24.3) and omitting the last term, which is canceled by mass renormalization, we get

$$Y \approx \tfrac{1}{2}[(\alpha\pi) + \alpha_m m - m - (\pi^2 + h)/2m] \times [\log(2g/m) + \tfrac{1}{2}] + h/4m.$$

Now

$$(\alpha\pi) + \alpha_m m - m - (\pi^2 + h)/2m = E - |E| \approx 0.$$

Thus

$$Y \approx h/4m = e\mathcal{H}\sigma_3/4m. \tag{24.5}$$

This corresponds to an extra energy

$$(e^2/\pi)Y_{ii} = e^3\mathcal{H}\sigma_3/4\pi m. \tag{24.6}$$

The elementary theory gives for the energy in the magnetic field $e\mathcal{H}\sigma_3/2m$. So the magnetic moment is increased by the factor $(1 + e^2/2\pi)$, or, bringing in the constants, $(1 + e^2/2\pi hc)$. This value for the anomalous magnetic moment of the electron is in good agreement with experiment.

Formula (24.5) contains a rather stronger result than we used in (24.6)—that Y doesn't have any nondiagonal elements. That means that the eigenfunctions are undisturbed by the interaction to the accuracy to which we have worked things out. The interaction in the case of a static field consisting of a uniform magnetic field is thus a very smooth thing which doesn't perturb the eigenfunctions of the energy. Its only effect is to give an extra energy term corresponding to the anomalous magnetic moment.

If you want to compare the theory I have given here with the usual theory working in terms of the Schrödinger picture, I think the best book to use is W. Heitler, *The Quantum Theory of Radiation*, 3rd ed., Oxford University Press, London and New York, 1954. If you make the comparison, you will find that the old work is not really a logical deduction such as we've been giving here, but still you have the same sort of integrals turning up, and the same substitutions have been made for evaluating those integrals. There's a great deal of similarity in the details of the calculations, but the argument of course is very different. I think the argument is a poor one when one tries to work in terms of the Schrödinger picture.

Calculations of the anomalous magnetic moment have been made to the fourth order, that is, to e^4/h^2c^2, by R. Karpus and N. M. Kroll, *Phys. Rev.* **77**, 536 (1950). These calculations are extremely heavy.

The Karpus and Kroll calculations contain some numerical mistakes. They have been amended by A. Sommerfeld, *Phys. Rev.* **107,** 328 (1957), and *Ann. Phys.* **5,** 26 (1958), and also by Petermann, *Helv. Phys. Acta*, **30,** 407. (1957).

§25. *The Lamb Shift: Preliminary*

I would like now to go over to the other main application of quantum electrodynamics to the Lamb shift. We now have a static electric field but no static magnetic field; that means we have a potential $\mathcal{A}_0$ which can be any function of x_1, x_2, x_3, $\mathcal{A}_0 = \mathcal{A}_0(x_1, x_2, x_3)$, but is not a function of time, and the static magnetic field is now zero, $\mathcal{A}_r = 0$.

That gives us for the one-electron energy problem $E = (\boldsymbol{\alpha}, \mathbf{p}) + \alpha_m m - e\mathcal{A}$, where $\mathcal{A}$ is written for $\mathcal{A}_0$, because we've only got one potential coming in this work and there's no need to write the suffix all the way through. I write, for short, $E = U - e\mathcal{A}$, where U stands for the operator $(\boldsymbol{\alpha}, \mathbf{p}) + \alpha_m m$, so $U^2 = {p_0}^2$.

We get $E^2 = {p_0}^2 - e(U\mathcal{A} + \mathcal{A}U)$, neglecting $\mathcal{A}^2$. $\mathcal{A}$ is a general function of x_1, x_2, x_3, so of course it does not commute with U; that is why we have to write these two terms $U\mathcal{A}$ and $\mathcal{A}U$ separately. We've now got to find the square root of E^2 to get $|E|$. How do we find the square root of a general operator like E^2? We can do it when we take $U\mathcal{A} + \mathcal{A}U$ small, and the result is

$$|E| = p_0 - \tfrac{1}{2}(UB + BU), \tag{25.1}$$

with a suitable quantity B of order $\mathcal{A}$. One can find out just what this quantity B is by squaring $|E|$ and comparing with E^2. If one does this, one sees that one needs

$$p_0 B + B p_0 = 2e\mathcal{A}. \tag{25.2}$$

Let us resolve $\mathcal{A}$ into Fourier components:

$$\mathcal{A}_x = \int \mathcal{A}_k e^{-i(\mathbf{k},\mathbf{x})}\, d^3k. \tag{25.3}$$

$\mathcal{A}_x$, of course, is just a number. I ought to point out that this Fourier resolution is essentially different from the Fourier resolution that we made for the electromagnetic field variables. Here there is no variation with time; it is a straightforward Fourier resolution in three dimensions. From (20.3) we find that

$$\langle p'|\mathcal{A}|p''\rangle = \int \mathcal{A}_k \, \delta_3(p' - p'' + k)\, d^3k = \mathcal{A}_{p''-p'} .$$

Now let us take (25.2) and express it in this matrix representation; we get

$$(p_0' + p_0'')\langle p'|B|p''\rangle = 2e\langle p'|\mathcal{A}|p''\rangle = 2e\mathcal{A}_{p''-p'} ,$$

where $p_0' = (m^2 + p'^2)^{1/2}, p_0'' = (m + p''^2)^{1/2}$. Well, that fixes B:

$$\langle p'|B|p''\rangle = \frac{2e\mathcal{A}_{p''-p'}}{p_0' + p_0''} . \tag{25.4}$$

Now from (25.1) $|E|^{-1} = (p_0)^{-1} + (2p_0)^{-1}(UB + BU)(p_0)^{-1}$. We have used the general formula $(x + n)^{-1} = x^{-1} - x^{-1}nx^{-1}$ for expanding a reciprocal with n small, when we have noncommuting quantities coming in. We now form $E/|E|$; E and $|E|$ commute with each other, so we don't need to specify an order for the product. If we evaluate the product with E on the left, we get

$$E/|E| = (U - e\mathcal{A})(p_0)^{-1} + (2p_0)^{-1}(p_0{}^2B + UBU)(p_0)^{-1},$$

with neglect of second-order quantities. Using (25.2), we get

$$\frac{E}{|E|} = \frac{U}{p_0} - \frac{B}{2} + \frac{1}{2}\frac{U}{p_0}B\frac{U}{p_0} = \frac{U}{p_0} - \frac{1}{p_0}N\frac{1}{p_0}, \tag{25.5}$$

where N is defined to be

$$N = \tfrac{1}{2}(p_0Bp_0 - UBU). \tag{25.6}$$

§26. *The Lamb Shift: Vacuum-Polarization Term*

I should now like to proceed to calculate the vacuum-polarization term. This term gave no contribution to the anomalous magnetic moment, but it does give a contribution to the Lamb shift and we have to work it out.

The formula for the vacuum-polarization term is (20.1). We only want the diagonal elements of Z for calculating an energy change, and

I shall confine myself to working out the diagonal elements because it's too complicated otherwise. They are

$$Z_{ii} = \pi^{-1} \int J_k^{\mu} \langle i | \alpha_{\mu} e^{-i(\mathbf{k},\mathbf{x})} | i \rangle \frac{d^3k}{|k|^2}. \tag{26.1}$$

J_k^{μ} is given by (20.2). We can evaluate the diagonal sum in any representation, and of course we now choose the momentum representation, as we already have our quantities A and B expressed in terms of this representation. We get

$$J_k^{\mu} = \tfrac{1}{4} \iint \langle p'' | e^{i(\mathbf{k},\mathbf{x})} | p' \rangle \langle p' | \left\langle \alpha^{\mu} \left(1 - \frac{E}{|E|} \right) \right\rangle | p'' \rangle \, d^3p' \, d^3p''.$$

Here we have taken the diagonal sum as far as it concerns momentum variables. We also have to take the diagonal sum with respect to spin variables; this is indicated by $\langle \alpha^{\mu}[1 - (E/|E|)] \rangle$. Using (20.3) and (25.4), we get

$$J_k^{\mu} = \tfrac{1}{4} \iint \langle p' | \left\langle \alpha^{\mu} \left[1 - \frac{U}{p_0} + \frac{1}{p_0} N \frac{1}{p_0} \right] \right\rangle | p'' \rangle$$
$$\times \, \delta(p' - p'' + k) \, d^3p' \, d^3p''.$$

The first two terms in the [ ] contribute nothing, because we take k to be different from zero. We are not interested in the case of $k = 0$; that would just correspond to a constant potential, which we've already dealt with in our discussion of cutoff (c); we need to take into account only those terms of [ ] which have a nonvanishing Fourier coefficient for $k \neq 0$.

Now let us go back to the expression (25.6) for N. It involves only even powers of α_1, α_2, α_3, α_m. It follows that $\langle \alpha_r N \rangle = 0$ $(r = 1, 2, 3)$, because $\alpha_r N$ will contain only odd powers of the α's, and so has no diagonal sum. Hence $J_k^{r} = 0$. We are left only with the problem of evaluating J_k^{0}:

$$J_k^{0} = \tfrac{1}{4} \iint \langle p' | (p_0)^{-1} \langle N \rangle (p_0)^{-1} | p'' \rangle \, \delta(p' - p'' + k) \, d^3p' \, d^3p''.$$

Now we need to evaluate $\langle N \rangle$. It is only those terms in N that don't contain any α matrices at all which contribute to the diagonal sum. So we have

$$\tfrac{1}{4} \langle N \rangle = \tfrac{1}{2}(p_0 B p_0 - p_r B p_r - m^2 B).$$

Substituting in J_k^μ and using (25.4), we have

$$J_k^0 = \iint \delta_3(p' - p'' + k) \frac{e\mathcal{a}_{p''-p'}}{p_0' + p_0''} \times \frac{p_0' p_0'' - p_r' p_r'' - m^2}{p_0' p_0''} d^3p' \, d^3p''.$$

It is convenient to go to new variables of integration. Put

$$p' = p - \tfrac{1}{2}k, \qquad p'' = p + \tfrac{1}{2}k,$$

so that $p'' - p' = k$. With this change we get

$$J_k^0 = e\mathcal{a}_k \int \frac{p_0' p_0'' - p^2 + \frac{1}{4}k^2 - m^2}{(p_0' + p_0'')p_0' p_0''} d^3p = e\mathcal{a}_k I_k \tag{26.2}$$

say.

If you compare this with Heitler's book, I_k is Heitler's L_{44},† with $k_0 = 0$. Using quite different arguments in terms of the Schrödinger picture, one turns up with the same integral in the end. The integral I_k is the thing that determines the vacuum-polarization contribution to the Lamb shift.

We have to evaluate the integral I_k and we can do it by making suitable substitutions. First of all, **k** is a constant as far as concerns this integration, so we can choose **k** in a special direction. Take **k** in the direction of the axis x_3, $k = 0, 0, k_3$, to simplify things; and put $p_1 = \rho \cos \phi$, $p_2 = \rho \sin \phi$. Thus

$$p_0' = (\rho^2 + p_3^2 + m^2 + \tfrac{1}{4}k_3^2 - p_3 k_3)^{1/2},$$
$$p_0'' = (\rho^2 + p_3^2 + m^2 + \tfrac{1}{4}k_3^2 + p_3 k_3)^{1/2}.$$

Now we are going to introduce a new variable by putting

$$p_0' + p_0'' = \omega.$$

You will note some analogy in this work to what we did in evaluating the integral in the case of the free electron. There again we introduced a variable ω which was the total energy.

We are going to pass to ρ, φ, ω as variables of integration. We have $d^3p = \rho \, d\rho \, d\phi \, dp_3$, and we have to work out the Jacobian

$$\frac{\partial(\omega, p_3)}{\partial(\rho, p_3)} = \left(\frac{\partial\omega}{\partial\rho}\right)_{p_3=\text{const}} = \frac{\rho\omega}{p_0' p_0''}.$$

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† W. Heitler, *The Quantum Theory of Radiation*, 3rd ed., Oxford University Press, London and New York, 1954, p. 322, eq. (16).

The result is that $d^3p/p_0'p_0'' = (d\omega/\omega)\,d\phi\,dp_3$. With this change of variables, we have

$$\begin{aligned} I_k &= 2\pi\int (p_0'p_0'' - p^2 + \tfrac{1}{4}k^2 - m^2)\omega^{-2}\,d\omega\,dp_3 \\ &= 2\pi\int [p_0'p_0'' - \tfrac{1}{2}(p_0'^2 + p_0''^2) + \tfrac{1}{2}k_3{}^2]\frac{d\omega}{\omega^2}\,dp_3 \\ &= \pi\int [k_3{}^2 - (p_0'' - p_0')^2]\frac{d\omega}{\omega^2}\,dp_3 . \end{aligned}$$

Now we note that

$$(p_0'' - p_0')\omega = p_0''^2 - p_0'^2 = 2p_3k_3,$$

and this enables us to write

$$I_k = \pi k_3{}^2\int\left(1 - \frac{4p_3{}^2}{\omega^2}\right)\frac{d\omega}{\omega^2}\,dp_3 .$$

Let us carry out the integration first with respect to p_3. We need to know the limits of integration for each value of ω. The limits of p_3 occur when $\rho = 0$, and then

$$\omega = [(p_3 + \tfrac{1}{2}k_3)^2 + m^2]^{1/2} + [(p_3 - \tfrac{1}{2}k_3)^2 + m^2]^{1/2}$$

and

$$\begin{aligned} \tfrac{1}{2}\omega^2 = p_3{}^2 + \tfrac{1}{4}k_3{}^2 + m^2 + [(p_3 + \tfrac{1}{2}k_3)^2 + m^2] \\ \times [(p_3 - \tfrac{1}{2}k_3)^2 + m^2]^{1/2}, \end{aligned}$$

so

$$(-\tfrac{1}{2}\omega^2 + p_3{}^2 + \tfrac{1}{4}k_3{}^2 + m^2)^2 = (p_3{}^2 + \tfrac{1}{4}k_3{}^2 + m^2)^2 - p_3{}^2k_3{}^2.$$

This reduces to the following:

$$p_3{}^2 = \frac{\omega^2(\omega^2 - 4m^2 - k_3{}^2)}{4(\omega^2 - k_3{}^2)},$$

and plus and minus the square root of this quantity gives us the limitations on p_3 for each value of ω.

We now find with those limits of integration,

$$\int dp_3 = \omega\left(\frac{\omega^2 - 4m^2 - k_3{}^2}{\omega^2 - k_3{}^2}\right)^{1/2}$$

and

$$\int p_3{}^2\,dp_3 = \tfrac{1}{12}\omega^3\left(\frac{\omega^2 - 4m^2 - k_3{}^2}{\omega^2 - k_3{}^2}\right)^{3/2}.$$

Thus

$$I_k = \pi k_3{}^2 \int \left[\left(\frac{\omega^2 - 4m^2 - k_3{}^2}{\omega^2 - k_3{}^2}\right)^{1/2} - \frac{1}{3}\left(\frac{\omega^2 - 4m^2 - k_3{}^2}{\omega^2 - k_3{}^2}\right)^{3/2}\right]\frac{d\omega}{\omega}.$$

At this stage in the work we can go back to the general k and replace $k_3{}^2$ by k^2. To carry the integration further we put $\omega^2 - k^2 = z^2$, and then I_k goes over to

$$I_k = \frac{2\pi k^2}{3}\int \frac{(z^2 + 2m^2)(z^2 - 4m^2)^{1/2}}{z^2(z^2 + k^2)}\,dz.$$

A further substitution is needed to get something one can integrate: $1 - (4m^2/z^2) = y^2$. Then, with a little bit of reduction, we get

$$I_k = \frac{4\pi m^2 k^2}{3}\int \frac{3 - y^2}{4m^2 + k^2 - k^2 y^2}\,\frac{y^2\,dy}{1 - y^2},$$

which can be integrated by standard methods.

Now what are the limits of integration of our variables. Well, ω^2 goes from $4m^2 + k^2$ to $4g^2$, so $4m^2 \le z^2 \le 4g^2$, and y goes from 0 to $1 - (m^2/g^2)$. We take the upper limit of ω to be $2g$, because we're using cutoff (c). Our procedure of using the variable of integration ω is really very suitable for working with cutoff (c), because ω is just the total energy of the intermediate state, which goes up to $2g$ if we neglect $|k|$ with respect to g.

Carrying out the integration, we get the result

$$I_k = \frac{4\pi m^2}{3}$$

$$\times \left\{2\gamma \log\frac{2g}{m} - 1 + (1 - 2\gamma)\left(\frac{\gamma + 1}{\gamma}\right)^{1/2} \log\left[\gamma^{1/2} + (\gamma + 1)^{1/2}\right]\right\}$$

where $\gamma = k^2/4m^2$. Now we have this rather awkward expression to deal with. Let us make the assumption that γ is small. You see what that means; it means that our electrostatic field does not have any components with very short wavelengths. That is quite reasonable when we apply it to the field in the hydrogen atom, for instance. Any wavelengths there will be long compared with m^{-1}, so the approximation is quite good.

We need to evaluate I_k to the accuracy of γ^2 and the second log to the accuracy $\gamma^{3/2}$. This log becomes $\gamma^{1/2} - \frac{1}{6}\gamma^{3/2} + \frac{3}{40}\gamma^{5/2}$, and the whole expression for I_k becomes

$$I_k = \frac{4\pi m^2}{3}\left[\left(2\log\frac{2g}{m} - \frac{5}{3}\right)\gamma - \frac{4}{5}\gamma^2\right] = qk^2 - \frac{\pi k^4}{15m^2},$$

with

$$q = \frac{2\pi}{3}\left(\log\frac{2g}{m} - \frac{5}{6}\right).$$

Equation (26.2) becomes

$$J_k{}^0 = \left(qk^2 - \frac{\pi k^4}{15m^2}\right)e\mathcal{A}_k.$$

Equation (26.1) becomes

$$\begin{aligned} Z_{ii} &= \frac{e}{\pi}\int\left(q - \frac{\pi k^2}{15m^2}\right)\mathcal{A}_k\langle i|e^{-i(k,x)}|i\rangle\, d^3k \\ &= \frac{e}{\pi}q\langle i|\int \mathcal{A}_k e^{-i(k,x)}\, d^3k|i\rangle - \frac{e}{15m^2}\langle i|\int k^2\mathcal{A}_k e^{-i(k,x)}\, d^3k|i\rangle \\ &= \frac{e}{\pi}q\langle i|\mathcal{A}_x|i\rangle + \frac{e}{15m^2}\langle i|\nabla^2\mathcal{A}_x|i\rangle, \qquad (26.3) \end{aligned}$$

with the help of (25.3). Z_{ii} multiplied by e^2/π gives a contribution to the energy.

Now you see that we have an infinity coming into the first term if we allow g to go to infinity, because we have $\log g$ coming into q. We can, however, get rid of this term by a procedure known as *charge renormalization*. It consists in assuming that the parameter e, which occurs in the Hamiltonian, is not the observed charge. We keep e to denote the physically observed charge and suppose that the charge parameter in the Hamiltonian, which we call $e + \delta e$, differs from the physically observed charge.

Now the Hamiltonian consists of two parts $H_0 + H_Q$. We are going to take δe of order e^3, so that adding the δe contribution to H_Q would not affect things to the order of accuracy to which we are working. We are only working to the order of accuracy of e^2 as far as the perturbation is concerned. But e also occurs in H_0 and δe makes a significant change in H_0, $\delta H_0 = -\delta e\int \mathcal{A}\bar{\psi}\psi\, d^3k$. Let us look upon this change as a further perturbation. We may write it

$$\delta H_0 = -\delta e\sum_{nn'}\bar{\psi}_n\langle n|\mathcal{A}|n'\rangle\psi_{n'}.$$

Making the transformation to the starred representation,

$$\delta H_0^* = -\delta e \sum_{nn'} \bar{\psi}_n \langle n|\mathcal{A}|n'\rangle \psi_{n'} e^{i(E_n - E_{n'})t}.$$

This gives

$$\left(i\frac{\partial K_2^*}{\partial t}\right)_{\text{extra}} = [\delta H_0^*, \bar{\psi}_i] = -\delta e \sum_{n} \bar{\psi}_n \langle n|\mathcal{A}|i\rangle e^{i(E_n - E_i)t}.$$

The term here for which $n = i$ also corresponds to a change in energy, and we get cancelation with the energy change coming from the first term of (26.3) by suitably choosing δe: $\delta e = (e^3/\pi^3)q$.

By changing the Hamiltonian in this way, a way that does not really have any physical significance and means simply that the charge parameter in the Hamiltonian differs from the observed charge, we are left with just one term in Z which gives a contribution to the Lamb shift.

Question: Must we get rid of that first term in Z_{ii} because it depends on g? Is that the reason?

Answer: I think that the answer to your question should be no. Even if q did not involve anything that became infinite with an infinite cutoff, charge renormalization would still be the proper thing to do. Otherwise, the energy associated with a charge e in a static field $\mathcal{A}_0$ would not be simply $e\mathcal{A}_0$; there would be an extra term and the energy would be $(e + \delta e)\mathcal{A}_0$. Now one usually takes the definition of the observed charge to be such that, if multiplied by $\mathcal{A}_0$, it gives the energy of the electron in a field $\mathcal{A}_0$. If one is going to work with that definition of the observed charge, one is forced to the conclusion that the observed charge is different from the e in the Hamiltonian.

§27. *The Lamb Shift: Relativistic Term*

I now want to calculate the other contribution to the Lamb shift, which comes from the integral Y:

$$Y = (4\pi)^{-1} \int \alpha_\mu \mathfrak{Y} \alpha^\mu \frac{d^3k}{|\mathbf{k}|}, \qquad \mathfrak{Y} = \frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\tilde{E}| + |\mathbf{k}| - \nu E_i}.$$

Let $\mathfrak{Y}_0$ be the corresponding operator with no external field $\mathcal{A}$ and with p_0 written for E_i, so

$$\mathfrak{Y}_0 = \frac{\nu + (\tilde{U}/\tilde{p}_0)}{\tilde{p}_0 + |\mathbf{k}| - \nu p_0}. \tag{27.1}$$

p_0 is by definition positive.

Now $\mathfrak{Y}_0$ gives an integral Y_0 which has already been worked out when we dealt with the case of no external field. The result was (22.3). When we bring in the usual mass renormalization we get an extra term

$$-\alpha_m m[\tfrac{3}{2}\log(2g/m) - \tfrac{1}{4}].$$

Adding this to (22.3), we get

$$Y_0 = \tfrac{1}{2}[(\alpha, \mathbf{p}) + \alpha_m m - p_0][\log(2g/m) + \tfrac{1}{2}].$$

Now we make use of weak equations that are appropriate to our present work. We have $E/|E| \approx 1$, which merely says we are starting off with a positive-energy state, and that leads to, from (25.5),

$$U - p_0 \approx N(p_0)^{-1}. \tag{27.2}$$

With $U = (\alpha, \mathbf{p}) + \alpha_m m$, we find that

$$Y_0 \approx \tfrac{1}{2}(\log(2g/m) + \tfrac{1}{2})N(p_0)^{-1}.$$

I shall use the notation $F \sim G$ to mean $\langle i|F|i\rangle = \langle i|G|i\rangle$. We only want to know the average value of the operator Y for the state $|i\rangle$. That average value is going to give us the Lamb shift. So we use this notation for things with the same average.

We have from (25.6),

$$\begin{aligned}\langle i|2N(p_0)^{-1}|i\rangle &= \langle i|(p_0B - UBU)(p_0)^{-1}|i\rangle = \langle i|p_0 - U)B|i\rangle,\\ &= -\langle i|(p_0)^{-1}NB|i\rangle\end{aligned}$$

from (27.2). Now, NB is of order $\mathcal{A}^2$ and is negligible. So

$$\frac{N}{p_0} \sim 0, \tag{27.3}$$

and Y_0 gives no contribution to the Lamb shift.

We must now evaluate $(4\pi)^{-1}\int \alpha_\mu(\mathfrak{Y} - \mathfrak{Y}_0)\alpha^\mu(d^3k/|\mathbf{k}|)$. We have got rid of the main part of the integral Y_0, and are left with quantities of the same order of magnitude as $\mathcal{A}$.

Let

$$D = (\text{denominator in } \tilde{\mathfrak{Y}}_0) = \tilde{p}_0 + |\mathbf{k}| - \nu p_0 .$$

Then

$$(\text{denominator in } \tilde{\mathfrak{Y}}) = D - \tfrac{1}{2}(\tilde{U}\tilde{B} + \tilde{B}\tilde{U}) + \nu(p_0 - E_i).$$

Now (den. in $\tilde{\mathfrak{Y}}$) $-$ D is weakly of order $\mathfrak{A}$. If we make an expansion counting (den. in $\tilde{\mathfrak{Y}}$) $-$ D as small we get

$$\tilde{\mathfrak{Y}} = \left(\nu + \frac{\tilde{E}}{|\tilde{E}|}\right)\{D^{-1} + D^{-1}[\tfrac{1}{2}(\tilde{U}\tilde{B} + \tilde{B}\tilde{U}) - \nu(p_0 - E_i)]D^{-1}\}. \tag{27.4}$$

I would like you to note that it is permissible to use the weak condition $p_0 - E_i \approx O(\mathfrak{A})$ here, because $(p_0 - E_i)$ commutes with D.

The expression (27.4) for $\tilde{\mathfrak{Y}}$ can be criticized because the denominator sometimes vanishes. Now we had corresponding trouble in the case of the calculation of the anomalous magnetic moment, but there the trouble was not serious, because it turned out that the region of k-space for which the denominator was small, and for which therefore the expansion was not valid, was such a small region that it did not affect the final result. However, for our present calculation that is no longer true. There is a region of k-space for which the expansion (27.4) is not valid. That region is too large for us to be able to use the expansion throughout the whole of k-space without getting into serious error. Under those conditions there is no method of approximation that is valid over the whole of k-space. What we have to do is split up our domain of k-space into two parts, the high-energy part and the low-energy part, and use the above expansion only for the high-energy part, in which it is legitimate, and use a different method of approximation for the low-energy part. It is a complicated procedure but it is the only way.

We separate the two domains of k by the condition that $|k| > \gamma m$ for the high-energy domain and $|\mathbf{k}| < \gamma m$ for the low-energy domain; and we have to choose γ suitably. Choose γ so that $\gamma \ll 1$, and at the same time $\gamma m \gg |e\mathfrak{A}|$. In the case of the hydrogen atom, where $|e\mathfrak{A}|$ is of the order of the binding energy, it is possible to choose γ in that way. γm has to be some energy between a few volts, the binding energy of the electron, and a half-million volts, corresponding to the rest mass of the electron. The precise value we give to γ does not matter, provided these inequalities are fulfilled. In this way we get two terms, which are called the *relativistic* (Y_{R}) and the *nonrelativistic* (Y_{NR}) contributions to the Lamb shift.

Let us proceed to calculate Y_{R}, using the expansion (27.4) and using (25.5) for $E/|E|$. We have the equation

$$\tfrac{1}{2}(UB + BU) = \alpha_m mB + \tfrac{1}{2}(\boldsymbol{\alpha}, \mathbf{p}B + B\mathbf{p}),$$

and from (25.2) and (25.6),

$$\frac{1}{2}\,\frac{U}{p_0}(UB + BU) = e\mathcal{A} - (p_0)^{-1}N.$$

These equations give us

$$\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_0 = \tilde{\mathcal{Y}}_1 + \tilde{\mathcal{Y}}_2 + \tilde{\mathcal{Y}}_3,$$

where

$$\tilde{\mathcal{Y}}_1 = \frac{\nu}{2D}(\boldsymbol{\alpha}, \tilde{\mathbf{p}}\tilde{B} + \tilde{B}\tilde{\mathbf{p}})D^{-1} - \frac{\nu(\boldsymbol{\alpha}, \tilde{\mathbf{p}})}{\tilde{p}_0 D^2}(p_0 - E_i), \tag{27.5}$$

$$\tilde{\mathcal{Y}}_2 = -(\tilde{p}_0)^{-1}(D^{-1}\tilde{N} + \tilde{N}(\tilde{p}_0)^{-1})D^{-1}, \tag{27.6}$$

$$\tilde{\mathcal{Y}}_3 = \frac{\nu}{D}\left[\alpha_m m\tilde{B} - \left(\nu + \frac{\alpha_m m}{\tilde{p}_0}\right)(p_0 - E_i) + \nu e\mathcal{A}\right]D^{-1}. \tag{27.7}$$

I shall assume that the momentum of the electron is small and retain terms of order of p^2 but neglect terms of order p^3. We have to go as far as the p^2 terms to get anything interesting. I shall let

$$\mu = (k^2 + m^2)^{1/2}.$$

This is a different μ from what we had in the magnetic-moment case. Then

$$\tilde{p}_0 = [\mu^2 + 2(\mathbf{p}, \mathbf{k}) + \mathbf{p}^2]^{1/2} = \mu\left[1 + \frac{(\mathbf{p}, \mathbf{k})}{\mu^2} - \frac{(\mathbf{p}, \mathbf{k})^2}{2\mu^4} + \frac{\mathbf{p}^2}{2\mu^2}\right], \tag{27.8}$$

$$\tilde{p}_0{}^{-1} = \mu^{-1}\left[1 - \frac{(\mathbf{p}, \mathbf{k})}{\mu^2} + \frac{3}{2}\,\frac{(\mathbf{p}, \mathbf{k})^2}{\mu^4} - \frac{\mathbf{p}^2}{2\mu^2}\right]. \tag{27.9}$$

We are now going to work with the cutoff (a), because the difference between the cutoffs (a) and (c) is negligible. It is important to use the right cutoff when calculating Y_0, but the difference between Y and Y_0 is small, and for this difference it does not matter which cutoff we use.

Now let us calculate $\mathfrak{Y}_1$. You should note that in the expansion of D^{-1} and in $\tilde{B}$ we can now neglect the p^2 terms. We need retain only the terms linear in p. The reason for that is that every term in $\mathfrak{Y}_1$ contains $(\boldsymbol{\alpha}, \tilde{\mathbf{p}})$, i.e., $(\boldsymbol{\alpha}, \mathbf{p})$ or $(\boldsymbol{\alpha}, \mathbf{k})$. Now if a term already contains p, any factors in that term need be evaluated only to the order of p, because we want the result to the accuracy of p^2. Also, if the term contains $(\boldsymbol{\alpha}, \mathbf{k})$ and is multiplied by a p^2 factor, it will give zero when we average it over all k directions.

We have from (25.4),

$$\langle p'|\tilde{B}|p''\rangle = \frac{2e\mathcal{A}_{p''-p'}}{\tilde{p}_0'' + \tilde{p}_0'},$$

where

$$\tilde{p}_0' = [m^2 + (\mathbf{p} + \mathbf{k})^2]^{1/2}, \qquad \tilde{p}_0'' = [m^2 + (\mathbf{p}'' + \mathbf{k})^2]^{1/2}.$$

Thus we have from (27.9),

$$\langle p'|\tilde{B}|p''\rangle = e\mathcal{A}_{p''-p'}\mu^{-1}\left[1 - \frac{(\mathbf{p}' + \mathbf{p}'', \mathbf{k})}{2\mu^2}\right].$$

Also we have

$$D = \tilde{p}_0 + |\mathbf{k}| - \nu p_0 = \mu + \frac{(\mathbf{p}, \mathbf{k})}{\mu} + |\mathbf{k}| - \nu m = \lambda + \frac{(\mathbf{p}, \mathbf{k})}{\mu},$$

where $\lambda = \mu + |\mathbf{k}| - \nu m$. So

$$D^{-1} = \lambda^{-1}\left[1 - \frac{(\mathbf{p}, \mathbf{k})}{\mu\lambda}\right].$$

There are the two terms for $\mathfrak{Y}_1$ in (27.5), which I shall deal with separately, say $\mathfrak{Y}_1 = \mathfrak{Y}_{11} + \mathfrak{Y}_{12}$. We have

$$\langle p'|\mathfrak{Y}_{11}|p''\rangle = \frac{\nu}{2D'D''}(\boldsymbol{\alpha}, \mathbf{p}' + \mathbf{p}'' + 2\mathbf{k})\frac{e\mathcal{A}_{p''-p'}}{\mu} \times\left[1 - \frac{(\mathbf{p}' + \mathbf{p}'', \mathbf{k})}{2\mu^2}\right],$$

which reduces after averaging over k directions to

$$\langle p'|\mathfrak{Y}_{11}|p''\rangle = \frac{\nu e\mathcal{A}_{p''-p'}}{2\mu\lambda^2}(\boldsymbol{\alpha}, \mathbf{p}' + \mathbf{p}'')\left(1 - \frac{k^2}{3\mu^2} - \frac{2k^2}{3\mu\lambda}\right).$$

We may write this result in symbolic form

$$\mathfrak{Y}_{11} = \frac{\nu e}{2\mu\lambda^2}(\boldsymbol{\alpha}, \mathbf{p}\mathcal{A} + \mathcal{A}\mathbf{p})\left(1 - \frac{k^2}{3\mu^2} - \frac{2k^2}{3\mu\lambda}\right).$$

Again

$$\begin{aligned}\tilde{\mathfrak{Y}}_{12} &= \frac{\nu}{\mu\lambda^2}(\boldsymbol{\alpha}, \mathbf{p} + \mathbf{k})\left(1 - \frac{2(\mathbf{p}, \mathbf{k})}{\mu\lambda}\right)\left(1 - \frac{(\mathbf{p}, \mathbf{k})}{\mu^2}\right)(E_i - p_0) \\ &= \frac{\nu}{\mu\lambda^2}(\boldsymbol{\alpha}, \mathbf{p})\left(1 - \frac{k^2}{3\mu^2} - \frac{2k^2}{3\mu\lambda}\right)(E_i - p_0).\end{aligned}$$

We add these two terms together and carry out the integration and the ν-averaging. We get

$$\begin{aligned}Y_1 &= (4\pi)^{-1} \int_{\gamma m}^{g} \alpha_\mu(\tilde{\mathfrak{Y}}_{11} + \tilde{\mathfrak{Y}}_{12})\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \\ &= m^{-1}\left(\log\frac{1}{2\gamma} - \frac{1}{2}\right)[(\boldsymbol{\alpha}, \mathbf{p})(E_i - p_0) + \tfrac{1}{2}e(\boldsymbol{\alpha}, \mathbf{p}\mathcal{A} + \mathcal{A}\mathbf{p})].\end{aligned}$$

We can reduce this somewhat by considering expressions that have the same average. First, we see that $(\boldsymbol{\alpha}, \mathbf{p})\mathcal{A} \sim (E_i - \alpha_m m)\mathcal{A}$ with neglect of second-order terms in $\mathcal{A}$. This has the same average for the state $|i\rangle$ as $\mathcal{A}(E_i - \alpha_m m) \sim \mathcal{A}(\boldsymbol{\alpha}, \mathbf{p})$. Thus we have

$$\begin{aligned}&(\boldsymbol{\alpha}, \mathbf{p})(E_i - p_0) + \tfrac{1}{2}e[(\boldsymbol{\alpha}, \mathbf{p})\mathcal{A} + \mathcal{A}(\boldsymbol{\alpha}, \mathbf{p})] \\ &\quad\sim (\boldsymbol{\alpha}, \mathbf{p})(E_i - p_0 + e\mathcal{A}) \\ &\quad\sim (\boldsymbol{\alpha}, \mathbf{p})(U - p_0) \sim (E_i - \alpha_m m)(U - p_0) \sim (E_i - \alpha_m m)N(p_0)^{-1} \\ &\quad\sim -\alpha_m m N(p_0)^{-1},\end{aligned}$$

where we have used (27.2) and (27.3) in the last two steps. We get the final result

$$Y_1 \sim -\log[(2\gamma)^{-1} - \tfrac{1}{2}]\alpha_m N(p_0)^{-1}. \tag{27.10}$$

Let us consider $\tilde{\mathfrak{Y}}_2$ next. We have from (25.6),

$$N = \tfrac{1}{2}[p_0 B p_0 - (\boldsymbol{\alpha}, \mathbf{p})B(\boldsymbol{\alpha}, \mathbf{p}) - m^2 B - (\boldsymbol{\alpha}, \mathbf{p})B\alpha_m m - \alpha_m m B(\boldsymbol{\alpha}, \mathbf{p})]. \tag{27.11}$$

Now use the formula, valid for any two vectors P, Q,

$$(\boldsymbol{\alpha}, \mathbf{P})(\boldsymbol{\alpha}, \mathbf{Q}) = (\mathbf{P}, \mathbf{Q}) + i(\boldsymbol{\sigma}, \mathbf{P}, \mathbf{Q}),$$

where $(\boldsymbol{\sigma}, \mathbf{P}, \mathbf{Q})$ is the triple scalar product. Taking the p', p'' matrix element of N, we get with the help of (25.4),

$$\begin{aligned}\langle p'|N|p''\rangle &= \frac{e\mathcal{A}_{p''-p'}}{p_0' + p_0''} \\ &\quad\times [p_0'p_0'' - (\mathbf{p}', \mathbf{p}'') - i(\boldsymbol{\sigma}, \mathbf{p}', \mathbf{p}'') - m^2 - (\boldsymbol{\alpha}, \mathbf{p}' - \mathbf{p}'')\alpha_m m].\end{aligned}$$

We are now going to use

$$\alpha_\mu \alpha^\mu = -2, \qquad \alpha_\mu \sigma_r \alpha^\mu = 2\sigma_r, \qquad \alpha_\mu \alpha_r \alpha_m \alpha^\mu = 0.$$

With these results we get

$$\langle p'|\alpha_\mu N \alpha^\mu|p''\rangle = -\frac{2e\mathcal{A}_{p''-p'}}{p_0' + p_0''} \times [p_0' p_0'' - (\mathbf{p}', \mathbf{p}'') + i(\sigma, \mathbf{p}', \mathbf{p}'') - m^2]. \quad (27.12)$$

Applying the $\sim$ transformation to both sides we get, after some reduction,

$$\langle p'|\alpha_\mu \tilde{N} \alpha^\mu|p''\rangle = -e\mathcal{A}_{p''-p'}\left[\tfrac{1}{2}(\mathbf{p}' - \mathbf{p}'')^2 - \frac{(\mathbf{p}' - \mathbf{p}'', \mathbf{k})^2}{2\mu^2} + i(\sigma, \mathbf{p}', \mathbf{p}'') + i(\sigma, \mathbf{p}' - \mathbf{p}'', \mathbf{k})\right]\mu^{-1} \times \left[1 - \frac{(\mathbf{p}' + \mathbf{p}'', \mathbf{k})}{2\mu^2}\right].$$

I would like you to note that in the small terms of order p^2 we can carry through the averaging over k directions already at this stage. It is not permissible to do that for the larger terms, because we shall later have to multiply this expression into other factors containing k. So we get

$$\langle p'|\alpha_\mu \tilde{N} \alpha^\mu|p''\rangle = -\frac{e}{\mu}\mathcal{A}_{p''-p'}\left\{[\tfrac{1}{2}(\mathbf{p}' - \mathbf{p}'')^2 + i(\sigma, \mathbf{p}', \mathbf{p}'')] \times \left(1 - \frac{\mathbf{k}^2}{3\mu^2}\right) + i(\sigma, \mathbf{p}' - \mathbf{p}'', \mathbf{k})\right\}.$$

Now we form

$$\langle p'|\alpha_\mu \widetilde{\mathcal{Y}}_2 \alpha^\mu|p''\rangle = \frac{e}{\mu}\mathcal{A}_{p''-p'}\left\{[\tfrac{1}{2}(\mathbf{p}' - \mathbf{p}'')^2 + i(\sigma, \mathbf{p}', \mathbf{p}'')] \times \left(1 - \frac{\mathbf{k}^2}{3\mu^2}\right) + i(\sigma, \mathbf{p}' - \mathbf{p}'', \mathbf{k})\right\} \times (\tilde{p}_0' D'')^{-1}[(D')^{-1} + (\tilde{p}_0'')^{-1}].$$

Working out the factors in the last line to the first order, we get

$$\langle p'|\alpha_\mu \widetilde{\mathcal{Y}}_2 \alpha^\mu|p''\rangle = (\cdots)(\mu\lambda)^{-1}\{\lambda^{-1} + \mu^{-1} - (\mathbf{p}' + \mathbf{p}'', \mathbf{k}) \times [(\mu^2\lambda)^{-1} + (\mu^3)^{-1} + (\mu\lambda^2)^{-1}]\}.$$

We have to average over all k directions and that leads to

$$\frac{e}{\mu^2\lambda}\,\mathcal{A}_{p''-p'}\Big\{\big[\tfrac{1}{2}(\mathbf{p}'-\mathbf{p}'')^2+i(\boldsymbol{\sigma},\mathbf{p}',\mathbf{p}'')\big]\left(1-\frac{\mathbf{k}^2}{3\mu^2}\right)\\ \times(\lambda^{-1}+\mu^{-1})-\tfrac{2}{3}i(\boldsymbol{\sigma},\mathbf{p}',\mathbf{p}'')\mathbf{k}^2\\ \times\big[(\mu^2\lambda)^{-1}+(\mu^3)^{-1}+(\mu\lambda^2)^{-1}\big]\Big\}\,.$$

The integration over all k-values between the appropriate limits gives

$$\langle p'|\,Y_2|p''\rangle = (4\pi)^{-1}\int_{\gamma m}^{g}\langle p'|\alpha_\mu\widetilde{\mathcal{Y}}_2\alpha^\mu|p''\rangle\frac{d^3k}{|\mathbf{k}|}\\ = \frac{e\mathcal{A}_{p''-p'}}{2m^2}\big[(\mathbf{p}'-\mathbf{p}'')^2\{\tfrac{1}{2}\log(2\gamma)^{-1}+\tfrac{1}{6}\log 2+\tfrac{5}{18}\}\\ +i(\boldsymbol{\sigma},\mathbf{p}',\mathbf{p}'')\log(2\gamma)^{-1}\big].$$

We may write that result symbolically as follows. We have

$$\langle p'|\nabla^2\mathcal{A}|p''\rangle = -(\mathbf{p}'-\mathbf{p}'')^2\mathcal{A}_{p''-p'}.$$

Also

$$\langle p'|\mathbf{grad}\,\mathcal{A}|p''\rangle = i(\mathbf{p}'-\mathbf{p}'')\mathcal{A}_{p''-p'},$$

so

$$\langle p'|(\boldsymbol{\sigma},\mathbf{grad}\,\mathcal{A},\mathbf{p})|p''\rangle = i(\boldsymbol{\sigma},\mathbf{p}',\mathbf{p}'')\mathcal{A}_{p''-p'}.$$

Thus we get the result that

$$Y_2 = -\frac{e}{2m^2}\nabla^2\mathcal{A}\big[\tfrac{1}{2}\log(2\gamma)^{-1}+\tfrac{1}{6}\log 2+\tfrac{5}{18}\big]\\ +\frac{e}{2m^2}\log(2\gamma)^{-1}(\boldsymbol{\sigma},\mathbf{grad}\,\mathcal{A},\mathbf{p}). \tag{27.13}$$

Now let us take $\widetilde{\mathcal{Y}}_3$ given by (27.7). We can write it

$$\widetilde{\mathcal{Y}}_3 = -D^{-2}\left(1+\frac{\nu\alpha_m m}{\tilde{p}_0}\right)(p_0-E_i-e\mathcal{A})\\ +\frac{\nu}{D}\alpha_m m(\tilde{B}D^{-1}-(D\tilde{p}_0)^{-1}e\mathcal{A})-\frac{e}{D}(D^{-1}\mathcal{A}-\mathcal{A}D^{-1})\\ = \widetilde{\mathcal{Y}}_{31}+\widetilde{\mathcal{Y}}_{32}+\widetilde{\mathcal{Y}}_{33}\,.$$

Now let us take the first term here. We have to multiply it by α_μ on the left and the right and we get from (22.1),

$$\alpha_\mu\widetilde{\mathcal{Y}}_{31}\alpha^\mu = \frac{2}{D^2}\left(1-\frac{2\nu\alpha_m m}{\tilde{p}_0}\right)(p_0-E_i-e\mathcal{A}).$$

Now we can use the weak equation

$$p_0 - E_i - e\mathcal{A} \approx p_0 - U.$$

We can use weak equations here because we have already multiplied by α_μ on left and right. With the help of the further weak equation (27.2) we get

$$\begin{aligned} \alpha_\mu \mathcal{Y}_{31} \alpha^\mu &\approx -\frac{2}{D^2}\left(1 - \frac{2\nu\alpha_m m}{\tilde{p}_0}\right) N(p_0)^{-1} \\ &= -\frac{2}{\lambda^2}\left(1 - \frac{2\nu\alpha_m m}{\mu}\right) N(p_0)^{-1}. \end{aligned} \tag{27.14}$$

I have here used the zero-order approximations $D = \lambda$ and $\tilde{p}_0 = \mu$. This is permissible because N is a very small quantity, of order $p\mathcal{A}$. If we kept higher-order terms, the next approximation would involve terms with $(\mathbf{p}, \mathbf{k})$ and these terms cancel when one averages over all k directions, and then the next terms with $(\mathbf{p}, \mathbf{k})^2$ will be too small, because they will give a contribution to (27.14) of order p^3. So (27.14) is valid after averaging over k directions and neglecting p^3 terms.

From (27.3) the first term in the parentheses in (27.14) contributes zero to the average and we are left with

$$\alpha_\mu \mathcal{Y}_{31} \alpha^\mu \sim \frac{4\nu}{\lambda^2 \mu} \alpha_m m N(p_0)^{-1}$$

and

$$Y_{31} = (4\pi)^{-1} \int_{\gamma m}^{g} \alpha_\mu \mathcal{Y}_{31} \alpha^\mu \frac{d^3k}{|\mathbf{k}|} \sim 2 \log{(2\gamma)^{-1}} \alpha_m N(p_0)^{-1}. \tag{27.15}$$

Let us go on to $\mathcal{Y}_{33}$. We have the equation

$$\langle i|p^2 e\mathcal{A}|i\rangle = \langle i|p^2(U - E_i)|i\rangle = \langle i|(U - E_i)p^2|i\rangle = \langle i|e\mathcal{A}p^2|i\rangle.$$

So $\mathcal{A}p^2 \sim p^2\mathcal{A}$.

Now let us consider the quantity D^{-1}:

$$D^{-1} = \lambda^{-1}\left[1 - \frac{(\mathbf{p}, \mathbf{k})}{\mu\lambda} + p^2 \text{ terms}\right]\cdot \tag{27.16}$$

By the same argument as before, we get $D^{-2}\mathcal{A} \sim \mathcal{A}D^{-2}$ after averaging over all k directions. Using this result we can write

$$\begin{aligned} \alpha_\mu \mathcal{Y}_{33} \alpha^\mu &= 2e(D^{-2}\mathcal{A} - D^{-1}\mathcal{A}D^{-1}) \\ &\sim e[D^{-2}\mathcal{A} - 2D^{-1}\mathcal{A}D^{-1} + \mathcal{A}D^{-2}]. \end{aligned}$$

We take matrix elements and get

$$\begin{aligned}\langle p'|\alpha_\mu\tilde{\mathcal{Y}}_{33}\alpha^\mu|p''\rangle &\sim e\mathcal{A}_{p''-p'}[(D')^{-2} - 2(D'D'')^{-1} + (D'')^{-2}]\\ &= e\mathcal{A}_{p''-p'}[(D')^{-1} - (D'')^{-1}]^2,\end{aligned}$$

and, retaining only the first-order terms in $[(D')^{-1} - (D'')^{-1})$ given by (27.16),

$$\langle p'|\alpha_\mu\tilde{\mathcal{Y}}_{33}\alpha^\mu|p''\rangle \sim e\mathcal{A}_{p''-p'}\frac{(\mathbf{p}' - \mathbf{p}'', \mathbf{k})^2}{\lambda^4\mu^2}.$$

If we average over k directions we get

$$\langle p'|\alpha_\mu\tilde{\mathcal{Y}}_{33}\alpha^\mu|p''\rangle \sim \frac{e}{3}\mathbf{k}^2\mathcal{A}_{p''-p'}\frac{(\mathbf{p}' - \mathbf{p}'')^2}{\lambda^4\mu^2},$$

so that

$$\alpha_\mu\tilde{\mathcal{Y}}_{33}\alpha^\mu \sim -\frac{e\mathbf{k}^2}{3\lambda^4\mu^2}\nabla^2\mathcal{A}.$$

Now we can integrate

$$\begin{aligned}Y_{33} = (4\pi)^{-1}\int_{\gamma m}^{g}\alpha_\mu\tilde{\mathcal{Y}}_{33}\alpha^\mu\frac{d^3\mathbf{k}}{|\mathbf{k}|} &\sim -\frac{e}{6m^2}\\ &\times [\log(2\gamma)^{-1} - \tfrac{1}{2}\log 2 - \tfrac{1}{4}]\nabla^2\mathcal{A}.\end{aligned} \tag{27.17}$$

Let us go on to $\tilde{\mathcal{Y}}_{32}$. As we did before, we can deduce

$$(D^2\tilde{p}_0)^{-1}\mathcal{A} \sim \mathcal{A}(D^2\tilde{p}_0)^{-1},$$

and so

$$\alpha_\mu\tilde{\mathcal{Y}}_{32}\alpha^\mu \sim 2\nu\alpha_m m[2D^{-1}\tilde{B}D^{-1} - (D^2\tilde{p}_0)^{-1}e\mathcal{A} - e\mathcal{A}(D^2\tilde{p}_0)^{-1})].$$

Taking matrix elements we have

$$\begin{aligned}&\langle p'|\alpha_\mu\tilde{\mathcal{Y}}_{32}\alpha^\mu|p''\rangle\\ &\quad\sim -2\nu\alpha_m me\mathcal{A}_{p''-p'}\\ &\quad\quad\times\left[(D'^2\tilde{p}_0')^{-1} - \frac{4}{(\tilde{p}_0' + \tilde{p}_0'')D'D''} + (D''^2\tilde{p}_0'')^{-1}\right]\\ &\quad= -\frac{2\nu\alpha_m me\mathcal{A}_{p''-p'}}{\tilde{p}_0' + \tilde{p}_0''}\\ &\quad\quad\times\{2[(D')^{-1} - (D'')^{-1}]^2 + (\tilde{p}_0'' - \tilde{p}_0')\\ &\quad\quad\times[(\tilde{p}_0'D'^2)^{-1} - (\tilde{p}_0''D''^2)^{-1}]\},\end{aligned}$$

and to second order in $(\mathbf{p}, \mathbf{k})$ we have

$$
\begin{aligned}
&-\frac{2\nu\alpha_m m e \mathcal{A}_{p''-p'}}{\tilde{p}_0' + \tilde{p}_0''} \\
&\quad\times \left\{\frac{2(\mathbf{p}' - \mathbf{p}'', \mathbf{k})^2}{\mu^2\lambda^4} + \frac{(\mathbf{p}' - \mathbf{p}'', \mathbf{k})}{\mu}\right. \\
&\quad\times \left.\left(\frac{(\mathbf{p}', \mathbf{k})}{\mu^3\lambda^2} + \frac{2(\mathbf{p}', \mathbf{k})}{\mu^2\lambda^3} - \frac{(\mathbf{p}'', \mathbf{k})}{\mu^3\lambda^2} - \frac{2(\mathbf{p}'', \mathbf{k})}{\mu^2\lambda^3}\right)\right\} \\
= &-\frac{\nu\alpha_m m e \mathcal{A}_{p''-p'}}{\mu}(\mathbf{p}' - \mathbf{p}'', \mathbf{k})^2 \\
&\quad\times\left(\frac{2}{\mu^2\lambda^4} + \frac{1}{\mu^4\lambda^2} + \frac{2}{\mu^3\lambda^3}\right).
\end{aligned}
$$

Averaging over k directions makes

$$
(\mathbf{p}' - \mathbf{p}'', \mathbf{k})^2 \rightarrow \tfrac{1}{3}(\mathbf{p}' - \mathbf{p}'')^2 k^2,
$$

so

$$
\alpha_\mu \tilde{\mathcal{Y}}_{32}\alpha^\mu \sim \nu\alpha_m m e \nabla^2\mathcal{A}\,\frac{k^2}{3}\left(\frac{2}{\mu^3\lambda^4} + \frac{1}{\mu^5\lambda^2} + \frac{2}{\mu^4\lambda^3}\right).
$$

We integrate and get

$$
Y_{32} = (4\pi)^{-1}\int_{\alpha_m}^{g} \alpha_\mu \tilde{\mathcal{Y}}_{32}\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \sim e\frac{\nabla^2\mathcal{A}}{3m^2}[\log(2\gamma)^{-1} - \tfrac{1}{6}] \qquad (27.18)
$$

Well, now that we have done all the integrations, it just remains to collect the terms (27.15), (27.17) and (27.18) together,

$$
\begin{aligned}
Y_3 = Y_{31} + Y_{32} + Y_{33} \sim \frac{e\nabla^2\mathcal{A}}{m^2}&[\tfrac{1}{6}\log(2\gamma)^{-1} + \tfrac{1}{12}\log 2 - \tfrac{1}{72}] \\
&+ 2\log(2\gamma)^{-1}\alpha_m N(p_0)^{-1}.
\end{aligned}
$$

Adding on Y_1 and Y_2, given by (27.10) and (27.13), we get

$$
\begin{aligned}
Y_R \sim -\frac{e\nabla^2\mathcal{A}}{m^2}&[\tfrac{1}{12}\log(2\gamma)^{-1} + \tfrac{11}{72}] + \frac{e}{2m^2}\log(2\gamma)^{-1}(\boldsymbol{\sigma}, \mathbf{grad}\,\mathcal{A}, \mathbf{p}) \\
&+ [\log(2\gamma)^{-1} + \tfrac{1}{2}]\alpha_m N(p_0)^{-1}.
\end{aligned}
$$

In the last term in Y_{R} the factor $(p_0)^{-1}$ can be replaced by m^{-1} without affecting the order of accuracy. This term can be connected with the first two terms. We have from (27.11),

$$
2N = p_0 B p_0 - (\boldsymbol{\alpha}, \mathbf{p})B(\boldsymbol{\alpha}, \mathbf{p}) - m^2 B + \alpha_m m[B(\boldsymbol{\alpha}, \mathbf{p}) - (\boldsymbol{\alpha}, \mathbf{p})B].
$$

Now

$$(\boldsymbol{\alpha}, \mathbf{p})B \sim (E_i - \alpha_m m)B \sim B(E_i - \alpha_m m) \sim B(\boldsymbol{\alpha}, \mathbf{p}),$$

so

$$\begin{aligned} 2\alpha_m m N &\sim [p_0 B p_0 - (\boldsymbol{\alpha}, \mathbf{p})B(\boldsymbol{\alpha}, \mathbf{p}) - m^2 B]\, \alpha_m m \\ &\sim [p_0 B p_0 - (\alpha, \mathbf{p})B(\boldsymbol{\alpha}, \mathbf{p}) - m^2 B][E_i - (\boldsymbol{\alpha}, \mathbf{p})] \\ &\sim [p_0 B p_0 - (\boldsymbol{\alpha}, \mathbf{p})B(\boldsymbol{\alpha}, \mathbf{p}) - m^2 B]m, \end{aligned}$$

as the $(\boldsymbol{\alpha}, \mathbf{p})$ in $[E_i - (\boldsymbol{\alpha}, \mathbf{p})]$ gives terms of order $\mathcal{A}p^3$ and is negligible. So

$$\begin{aligned} 2\langle p'|\alpha_m N|p''\rangle &\sim \frac{2e\mathcal{A}_{p''-p'}}{p_0' + p_0''}[p_0' p_0'' - (\boldsymbol{\alpha}, \mathbf{p}')(\boldsymbol{\alpha}, \mathbf{p}'') - m^2] \\ &\sim \frac{e\mathcal{A}_{p''-p'}}{m}[\tfrac{1}{2}(\mathbf{p}' - \mathbf{p}'')^2 - i(\sigma, \mathbf{p}', \mathbf{p}'')]. \end{aligned}$$

So

$$\alpha_m N \sim -\frac{e}{4m}\nabla^2\mathcal{A} - \frac{e}{2m}(\sigma, \mathbf{grad}\ \mathcal{A}, \mathbf{p}).$$

Using this result we have

$$Y_R \sim -\frac{e}{3m^2}\nabla^2\mathcal{A}[\log\,(2\gamma)^{-1} + \tfrac{5}{6}] - \frac{e}{4m^2}(\sigma, \mathbf{grad}\ \mathcal{A}, \mathbf{p}). \qquad (27.19)$$

That is the final formula for the relativistic part of Y. It has these two terms, one of them involving $\nabla^2\mathcal{A}$, the other involving the electron spin.

You note the appearance of γ in this formula; we can't take γ equal to zero, so we have to make a cutoff for low values of k.

§28. *The Lamb Shift: Nonrelativistic Term*

Here we use quite a different approximation, an approximation that is appropriate to nonrelativistic electron theory, because here we are using k values that are small compared with the rest energy of the electron.

The nonrelativistic part of Y is defined as

$$Y_{NR} = (4\pi)^{-1}\int_0^{\gamma m} \alpha_\mu(\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_0)\alpha^\mu \frac{d^3k}{|\mathbf{k}|},$$

with

$$\tilde{\mathcal{Y}} = \frac{\nu + (\tilde{E}/|\tilde{E}|)}{|\tilde{E}| + |\mathbf{k}| - \nu E_i}, \qquad \tilde{\mathcal{Y}}_0 = \frac{\nu + (\tilde{U}/\tilde{p}_0)}{\tilde{p}_0 + |\mathbf{k}| - \nu p_0}.$$

The first thing to note is that we get small denominators only for $\nu = 1$. With regard to the terms when $\nu = -1$, there are no small denominators. We might have integrated the relativistic part for those terms right down to $k = 0$. The contribution of the terms with $\nu = -1$ to the nonrelativistic part is negligible. So instead of having to average over both values of ν, we just put $\nu = 1$ and put in a factor $\frac{1}{2}$. Thus we take

$$\mathcal{Y} = \frac{1}{2}\,\frac{1 + (\tilde{E}/|\tilde{E}|)}{|\tilde{E}| + |\mathbf{k}| - E_i}, \qquad \mathcal{Y}_0 = \frac{1}{2}\,\frac{1 + (\tilde{U}/\tilde{p}_0)}{\tilde{p}_0 + |\mathbf{k}| - p_0}.$$

Now, from (25.5) and (27.12) we know that

$$\alpha_\mu \left(\frac{E}{|E|} - \frac{U}{p_0} \right) \alpha^\mu = O(\mathcal{A}p^2).$$

There is not very much difference between the $\sim$ quantities and non-$\sim$ quantities now, because k is small; the difference between them is of order k only. We can thus replace $\tilde{U}/\tilde{p}_0$ by $\tilde{E}/|\tilde{E}|$ in $\mathcal{Y}_0$ without disturbing the accuracy of our calculations, the error being of order $\mathcal{A}p^2/|\mathbf{k}|$ in $\alpha_\mu \mathcal{Y}_0 \alpha^\mu$, giving an error in Y_{NR} of order

$$p^2 \mathcal{A} \int_0^{\gamma m} |\mathbf{k}|^{-1} \frac{d^3k}{|\mathbf{k}|} = 4\pi\gamma p^2 \mathcal{A},$$

which is negligible because we are taking $\gamma \ll 1$.

So now

$$\alpha_\mu(\mathcal{Y} - \mathcal{Y}_0)\alpha^\mu = \alpha_\mu \tfrac{1}{2}\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right) \times [(|\mathbf{k}| + |\tilde{E}| - E_i)^{-1} - (|\mathbf{k}| + \tilde{p}_0 - p_0)^{-1}]\,\alpha^\mu.$$

Expanding in $|\mathbf{k}|$, the above expression becomes

$$\frac{\alpha_\mu}{2}\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right) \times \left[\frac{1}{|\mathbf{k}|} - \frac{|\tilde{E}| - E_i}{|\mathbf{k}|(|\mathbf{k}| + |\tilde{E}| - E_i)} - \frac{1}{|\mathbf{k}|} + \frac{\tilde{p}_0 - p_0}{|\mathbf{k}|(|\mathbf{k}| + \tilde{p}_0 - p_0)}\right]\alpha^\mu. \tag{28.1}$$

For $\mu = 1, 2, 3$ we have α_r on the left and right, and the matrix elements of the velocity α_r are of the same order of magnitude as p/m

in a nonrelativistic theory. To see this we write

$$\begin{aligned}\langle i|\alpha_r|n\rangle &= (E_i + E_n)^{-1}\langle i|E\alpha_r + \alpha_r E|n\rangle \\ &= m^{-1}\langle i|p_r - e\mathcal{A}\alpha_r|n\rangle = m^{-1}\langle i|p_r|n\rangle,\end{aligned} \qquad (28.2)$$

because

$$\langle i|e\mathcal{A}\alpha_r|n\rangle \ll m\langle i|\alpha_r|n\rangle.$$

So we see that the velocity has approximately the same matrix elements as the momentum divided by the mass. The momentum is counted as small in this work and so for $\mu = 1, 2, 3$ we have a small momentum occurring to the right and left. The last term in the [ ] in (28.1) is then negligible and we are left with

$$\alpha_r(\mathcal{Y} - \mathcal{Y}_0)\alpha^r = \tfrac{1}{2}\alpha^r\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right)\frac{|\tilde{E}| - E_i}{|\mathbf{k}|(|\mathbf{k}| + |\tilde{E}| - E_i)}\alpha_r. \qquad (28.3)$$

For $\mu = 0$ we do not get this small momentum, so the term for $\mu = 0$ is substantially larger than the other three terms, and we have to calculate it with higher accuracy. To do this we use

$$\begin{aligned}\mathcal{Y} - \mathcal{Y}_0 = \frac{1}{2}\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right)\Bigg[&\frac{1}{|\mathbf{k}|} - \frac{|\tilde{E}| - E_i}{|\mathbf{k}|^2} + \frac{(|\tilde{E}| - E_i)^2}{|\mathbf{k}|^3} \\ &- \frac{(|\tilde{E}| - E_i)^3}{|\mathbf{k}|^3(|\mathbf{k}| + |\tilde{E}| - E_i} - \frac{1}{|\mathbf{k}|} + \frac{\tilde{p}_0 - p_0}{|\mathbf{k}|^2} \\ &- \frac{(\tilde{p}_0 - p_0)^2}{|\mathbf{k}|^3} + \frac{(\tilde{p}_0 - p_0)^3}{|\mathbf{k}|^3(|\mathbf{k}| + \tilde{p}_0 - p_0)}\Bigg].\end{aligned}$$

The term in $(\tilde{p}_0 - p_0)^3$ is negligible because it involves p^3. Also one can show that

$$(\tilde{p}_0 - p_0) - (|\tilde{E}| - E_i) \qquad \text{and} \qquad (\tilde{p}_0 - p_0)^2 - (|\tilde{E}| - E_i)^2$$

make negligible contributions. To prove that, we see from (25.1) that

$$\begin{aligned}(|\tilde{E}| - E_i) - (\tilde{p}_0 - p_0) &\sim -\tfrac{1}{2}(\tilde{U}\tilde{B} + \tilde{B}\tilde{U}) + \tfrac{1}{2}(UB + BU) \\ &= O(\mathcal{A}k).\end{aligned}$$

When multiplied by $[1 + (\tilde{E}/|\tilde{E}|)]$ and averaged over all k directions, this gives a quantity of order $\mathcal{A}k^2$. We must then multiply by $|\mathbf{k}|^{-2}(d^3k/|\mathbf{k}|)$ and integrate, so we get a term of order

$$\mathcal{A}\int_0^{\gamma_m} |\mathbf{k}|\, d|\mathbf{k}| = O(\gamma^2\mathcal{A}),$$

which is negligible. We can carry through a similar argument to show that $(\tilde{p}_0 - p_0)^2 - (|\tilde{E}| - E_i)^2$ gives a term of order $\gamma\mathcal{A}$ and is also negligible.

Now we are left with

$$\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0 \sim -\frac{1}{2}\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right)\frac{(|\tilde{E}| - E_i)^3}{|\mathbf{k}|^3(|\mathbf{k}| + |\tilde{E}| - E_i)}$$

$$= -\tfrac{1}{2}(|\tilde{E}| - E_i)\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right)$$

$$\times \frac{|\tilde{E}| - E_i}{|\mathbf{k}|^3(|\mathbf{k}| + |\tilde{E}| - E_i)}(|\tilde{E}| - E_i). \qquad (28.4)$$

All that I have done here is to rearrange the factors, and we can do that because all the factors commute with each other. The only things we have here that are not numbers are $\tilde{E}$ and $|\tilde{E}|$, and they commute with each other.

We may replace $|\tilde{E}|$ by $\tilde{E}$ in the factor on the right of (28.4) because we have got the factor $1 + (\tilde{E}/|\tilde{E}|)$, which secures that $\tilde{E} = |\tilde{E}|$, because when $\tilde{E} = -|\tilde{E}|$ it gives zero. We thus get on the right of (28.4) the factor

$$\tilde{E} - E_i = (\boldsymbol{\alpha}, \mathbf{p} + \mathbf{k}) + \alpha_m m - e\mathcal{A} - E_i \approx (\boldsymbol{\alpha}, \mathbf{k}).$$

When we take averages for the state i, we can treat the factor $(|\tilde{E}| - E_i)$ on the left of (28.4) in the same way, so that it also becomes $(\boldsymbol{\alpha}, \mathbf{k})$. So the term (28.1) for $\mu = 0$ has the same average as

$$-\tfrac{1}{2}(\boldsymbol{\alpha}, \mathbf{k})\left(1 + \frac{\tilde{E}}{|\tilde{E}|}\right)\frac{\tilde{E} - E_i}{|\mathbf{k}|^3(|\mathbf{k}| + |\tilde{E}| - E_i)}(\boldsymbol{\alpha}, \mathbf{k}). \qquad (28.5)$$

In the expressions (28.3) and (28.5) we can drop the $\sim$'s, because the α_r are already small (their matrix elements are small) and the difference between the $\sim$ and non-$\sim$ quantities involves k, which is also small. When we now average (28.5) over all k directions we get

$$\tilde{\mathfrak{Y}} - \tilde{\mathfrak{Y}}_0 \sim -\tfrac{1}{6}\alpha_r\left(1 + \frac{E}{|E|}\right)\left(\frac{|E| - E_i}{|\mathbf{k}|(|\mathbf{k}| + |E| - E_i)}\right)\alpha_r .$$

Adding on (28.3), we get

$$\alpha_\mu(\tilde{\mathfrak{Y}} - (\tilde{\mathfrak{Y}}_0)\alpha^\mu \sim \tfrac{1}{3}\alpha_r\left(1 + \frac{E}{|E|}\right)\frac{|E| - E_i}{|\mathbf{k}|(|\mathbf{k}| + |E| - E_i)}\alpha_r$$

Thus

$$Y_{\mathrm{NR}} = (4\pi)^{-1}\int_0^{\gamma m} \alpha_\mu(\mathfrak{Y} - \mathfrak{Y}_0)\alpha^\mu \frac{d^3k}{|\mathbf{k}|} \sim \tfrac{1}{3}\int_0^{\gamma m} \alpha_r \left(1 + \frac{E}{|E|}\right)$$
$$\times \frac{|E| - E_i}{|\mathbf{k}| + |E| - E_i}\,\alpha_r\, d|\mathbf{k}|.$$

We carry out the integration and get

$$Y_{\mathrm{NR}} \sim \tfrac{1}{3}\alpha_r\left(1 + \frac{E}{|E|}\right)(|E| - E_i)\log\frac{\gamma m + |E| - E_i}{\big||E| - E_i\big|}\,\alpha_r.$$

Now with sufficient accuracy we can neglect $|E| - E_i$ in the numerator of the log, because γm is large compared with the energy differences that we are interested in, so

$$\langle i|\, Y_{\mathrm{NR}}|i\rangle = \frac{2}{3}\sum_{En>0}\langle i|\alpha_r|n\rangle(E_n - E_i)\log\frac{\gamma m}{|E_n - E_i|}\langle n|\alpha_r|i\rangle$$
$$= \frac{2}{3}\sum_{En>0}\frac{E_n - E_i}{m^2}|\langle i|p_r|n\rangle|^2\log\frac{\gamma m}{|E_n - E_i|},$$

from (28.2).

This expression must be evaluated by computation for the particular problem in which one is interested. The result can be expressed in the form

$$\langle i|\, Y_{\mathrm{NR}}|i\rangle = \frac{2}{3}\sum_{En>0}\frac{E_n - E_i}{m^2}|\langle i|p_r|n\rangle|^2\log\frac{\gamma m}{M},$$

where M is some average of the $|E_n - E_i|$, which must be obtained from the computation. The computation must, of course, take into account the values of the matrix elements that multiply the log term; for various n values. M occurs only in the logarithm, so variations in M are not too important. Heitler† gives the value $\log(m/M) = 7.6876$ for the Lamb shift in hydrogen with the main quantum number 2.

The restriction $E_n > 0$ is not important, as the terms for $E_n < 0$ would contribute only a negligible amount, because of the smallness of the matrix elements for $E_n < 0$. Dropping this restriction, we can express the result in a more suitable form:

---

† W. Heitler, *The Quantum Theory of Radiation*, 3rd ed., Oxford University Press, London and New York, 1954, p. 346.

$$
\begin{aligned}
\langle i|\,Y_{\mathrm{NR}}|i\rangle &= \frac{2}{3m^2}\log\frac{\gamma m}{M}\,\langle i|p_r(E - E_i)p_r|i\rangle \\
&= \frac{2}{3m^2}\log\frac{\gamma m}{M}\,\langle i|p_r(U - e\mathcal{A} - E_i)p_r|i\rangle \\
&= \frac{1}{3m^2}\log\frac{\gamma m}{M}\,\langle i|(U - E_i)p_r^{\,2} - 2p_r e\mathcal{A}p_r + p_r^{\,2}(U - E_i)|i\rangle \\
&= \frac{1}{3m^2}\log\frac{\gamma m}{M}\,\langle i|e\mathcal{A}p_r^{\,2} - 2p_r e\mathcal{A}p_r + p_r^{\,2}e\mathcal{A}|i\rangle \\
&= \frac{-e}{3m^2}\log\frac{\gamma m}{M}\,\langle i|\nabla^2\mathcal{A}|i\rangle. \qquad (28.6)
\end{aligned}
$$

§29. *The Lamb Shift: Total*

So finally, adding the expressions (27.19) and (28.6) and the second term of (26.3), we get

$$
\begin{aligned}
\langle i|\,Y|i\rangle + \langle i|Z|i\rangle = &-\frac{e}{3m^2}\langle i|\nabla^2\mathcal{A}|i\rangle\left(\log\frac{m}{2M} + \frac{19}{30}\right) \\
&- \frac{e^2}{4m^2}\langle i|(\boldsymbol{\sigma},\, \mathbf{grad}\,\mathcal{A},\, \mathbf{p})|i\rangle.
\end{aligned}
$$

and

$$\frac{e^2}{\pi}\langle i|\,Y + Z|i\rangle = \text{Lamb shift.}$$

The important thing to note about this expression is that γ has disappeared from it. Of course, γ has to disappear if we have not made a slip anywhere, because the precise value at which we make the distinction between the relativistic and nonrelativistic parts is unimportant. γ must be of the right order of magnitude for the approximations to be valid, but just where we chose γ doesn't matter.

Well, that gives the calculation of the Lamb shift up to a certain degree of accuracy. People have carried the calculations through with higher accuracy, working in the Schrödinger picture. Professor Lamb told me recently that he is now doing experiments again, with a different apparatus, and hopes to improve the measurement of the Lamb shift by quite a substantial factor, and then it will be well worthwhile to have the theory of the Lamb shift worked out to a greater degree of accuracy, to compare it with these expected experimental results. Of course, the calculation will be very complicated, because it involves working to order $(e^2/\hbar c)^2$ or maybe even to $(e^2/\hbar c)^3$ to get the calculations with the high degree of accuracy with which the experiments will be performed. Also, we shall have to stop doing the other approximations we have made, counting the momentum as small. In all our work we have counted the momentum of the particle as small and worked only as far as p^2. It will be necessary work as far as p^4, and it may be necessary to work as far as the square of the potential $\mathfrak{A}$, so that all together there is a lot more work to be done to get the calculation of the Lamb shift to the desired degree of accuracy. The treatment I have given shows the method and shows that one is not bothered by infinities if one carries through the calculation correctly.

In our calculations of the anomalous magnetic moment and the Lamb shift we were dealing with one-electron problems, and the essential point in these calculations is to count the interaction energy as small. We counted $\int A_\mu j^\mu \, d^3x$ as small, and it seems that this is a good approximation for one-electron problems.

In a way this is rather surprising, because the quantity we are assuming to be small is not gauge-invariant. Our whole Hamiltonian is a gauge-invariant quantity, but we split it up into two parts, a proper-energy part and an interaction-energy part, and those two parts by themselves are not gauge-invariant. It is not altogether satisfactory to do that sort of thing, but it does seem to work for one-electron problems. One would like to have a calculation that works entirely with gauge-invariant quantities; if one could do that, the theory would be on a more satisfactory basis.

Another thing you should note about these calculations is that we have not made any use of the weak equations

$$\frac{\partial A_\mu}{\partial x^\mu} \approx 0. \qquad (29.1)$$

It seems that these weak equations do not play an important role in one-electron problems.

§30. *The Coulomb Force*

If we proceed to problems with more than one electron, the situation is quite different. If we have more than one electron present, it is not permissible to count H_Q as small, because we have Coulomb interactions coming into play. Then we have to make use of the constraint equations (29.1).

I should like to show you how the Coulomb interaction does come in. It involves a certain treatment of the longitudinal electromagnetic waves in the Hamiltonian. This work is given in all the books on the subject in connection with the Schrödinger picture, and it is called the transformation from the Lorentz gauge to the Coulomb gauge. I should like to show you that one can do corresponding work with the Heisenberg picture.

The constraints (29.1) give us two conditions on the Hamiltonian variables at a particular time, (16.6) and (16.7). Of course, there is one of these constraint equations for each point in three-dimensional space.

Let us pass to the Fourier components. We write

$$\begin{aligned} A_x^{\mu} &= \int (A_k^{\mu} + \bar{A}_{-k}^{\mu}) e^{-i(\mathbf{k},\mathbf{x})}\, d^3k, \\ B_x^{\mu} &= i\int |\mathbf{k}|(A_k^{\mu} - \bar{A}_{-k}^{\mu}) e^{-i(\mathbf{k},\mathbf{x})}\, d^3k, \\ j_{0x} &= \int j_{0k} e^{-i(\mathbf{k},\mathbf{x})}\, d^3k. \end{aligned} \tag{30.1}$$

Equations (16.6) and (16.7) written in terms of the Fourier coefficients are

$$i|\mathbf{k}|(A_{0k} - \bar{A}_{0-k}) - ik_r(A_{rk} + \bar{A}_{r-k}) \approx 0, \tag{30.2}$$

$$|\mathbf{k}|k_r(A_{rk} - \bar{A}_{r-k}) - |\mathbf{k}|^2(A_{0k} + \bar{A}_{0-k}) \approx -4\pi j_{0k}. \tag{30.3}$$

Take—[equation (30.3)]—$i|k|$ [equation (30.2)], and that gives us

$$2|\mathbf{k}|^2 A_{0k} - 2|\mathbf{k}|k_r A_{rk} \approx 4\pi j_{0k}. \tag{30.4}$$

Let us put $k_0 = |\mathbf{k}|$; then (30.4) becomes

$$k^{\mu} A_{\mu k} \approx \frac{2\pi}{k_0} j_{0k}. \tag{30.5}$$

The conjugate complex equation is

$$k^{\mu} \bar{A}_{\mu k} \approx \frac{2\pi}{k_0} j_{0-k}. \tag{30.6}$$

Now we split up the electromagnetic field into longitudinal and transverse parts and get an expression for the longitudinal part of the field energy. We put

$$A_{rk} = A^L_{rk} + A^T_{rk},$$

where
$A^L_{rk} = (k_r/k_0{}^2)k_sA_{sk}$ by definition, and $A^T_{rk} = A_{rk} - A^L_{rk}$. We have $k_rA^L_{rk} = k_sA_{sk}$, and therefore $k_rA^T_{rk} = 0$, which checks that the definition of the longitudinal part is correct.

Let us go back to our expression for the electromagnetic field energy expressed in terms of the Fourier components, (15.16), and ignore the infinite constant, which means fixing it so that there is no zero-point energy. We can write it $H_F = H_{FL} + H_{FT}$, with

$$H_{FL} = 4\pi^2\int k_0{}^2(A^L_{rk}\overline{A^L_{rk}} - \overline{A}_{0k}A_{0k})\,d^3k,$$

$$H_{FT} = 4\pi^2\int k_0{}^2A^T_{rk}\overline{A^T_{rk}}\,d^3k.$$

The cross terms containing $A^L_{rk}\overline{A^T_{rk}}$ cancel out. This simply means that there is no interference energy between the longitudinal and transverse waves of the field.

We can write H_{FL} as

$$\begin{aligned} H_{FL} = 4\pi^2\int &[k_rA_{rk}(k_0\overline{A}_{0k} - k^\mu\overline{A}_{\mu k}) \\ &-k_0\overline{A}_{0k}(k^\mu A_{\mu k} + k_rA_{rk})]\,d^3k \\ = 4\pi^2\int &[-(k_0A_{0k} - k^\nu A_{\nu k})k^\mu\overline{A}_{\mu k} - k_0\overline{A}_{0k}k^\mu A_{\mu k}]\,d^3k. \end{aligned} \tag{30.7}$$

Now we have to make use of the constraint equations. I discussed in Section 14 how we are to use constraints in the Heisenberg formulation; we have the rule that in general we can multiply constraints only by quantities on the left and not by quantities on the right. For the present work we find that we can get all the results we want if we multiply the constraint equations only by quantities that commute with them. Of course, in that case it is permissible to multiply them on either side.

We get in this way, from (30.5) and (30.6),

$$(A_{0k} + \bar{A}_{0k})k^\mu(A_{\mu k} + \bar{A}_{\mu k}) \approx \frac{2\pi}{k_0}(A_{0k} + \bar{A}_{0k})(j_{0k} + j_{0-k})$$

and

$$(A_{0k} - \bar{A}_{0k})k^\mu(A_{\mu k} - \bar{A}_{\mu k}) \approx \frac{2\pi}{k_0}(A_{0k} - \bar{A}_{0k})(j_{0k} - j_{0-k}).$$

Subtracting

$$A_{0k}k^\mu\bar{A}_{\mu k} + \bar{A}_{0k}k^\mu A_{\mu k} \approx \frac{2\pi}{k_0}(A_{0k}j_{0-k} + \bar{A}_{0k}j_{0k}).$$

Also, since $k^\nu A_{\nu k}$ and $k^\mu\bar{A}_{\mu k}$ commute, we can infer from (30.5) and (30.6),

$$k^\nu A_{\nu k}k^\mu\bar{A}_{\mu k} \approx \frac{4\pi^2}{{k_0}^2} j_{0k}j_{0-k}\,.$$

Using these results we find from (30.7),

$$H_{FL} \approx -8\pi^3\int(A_{0k}j_{0-k} + \bar{A}_{0k}j_{0k})\,d^3k + 16\pi^4\int\frac{j_{0k}j_{0-k}}{{k_0}^2}\,d^3k.$$

Then if we go back from the Fourier components to the field quantities, using (30.1), we get

$$H_{FL} = -\int A_{0x}j_{0x}\,d^3x + \frac{1}{2}\iint\frac{j_{0x}j_{0x'}}{|x - x'|}\,d^3x\,d^3x'.$$

The term $-\int A_{0x}j_{0x}\,d^3x$ cancels with one of the terms in the interaction energy H_Q, and

$$\frac{1}{2}\iint\frac{j_{0x}j_{0x'}}{|x - x'|}\,d^3x\,d^3x'$$

is the Coulomb interaction energy between any particles that are present. In that way the Coulomb interaction energy between the charged particles appears from the longitudinal part of the field energy.

The important point of this calculation is that you can carry it through in the Heisenberg picture using the constraint equations only to the extent that you multiply them by quantities that commute with them.

§31. *The General Physical Interpretation*

Another problem I would like to consider now is, what is the general physical interpretation for this theory? In the usual quantum mechanics one takes the Schrödinger wave function, normalizes it, and forms the square of its modulus and interprets what one gets as a probability. We cannot do that anymore because we have not got a Schrödinger wave function in this quantum field theory. What can we do instead?

In our calculation of the analomous magnetic moment and the Lamb shift, we had a normal ordering of our q-numbers. To make this normal ordering appear naturally in the calculations, one should introduce a certain ket vector at each time. At each time t, introduce a ket, which we may call the vacuum ket at time t and denote by $|O_t\rangle$, satisfying

$$(\text{any absorption operator at time } t)\,|O_t\rangle = 0.$$

This $|O_t\rangle$ would seem to play the role of the vacuum ket, but of course it differs very much from what one usually understands by the vacuum ket, because it is only an instantaneous thing; the ket that satisfies those conditions at time t would be quite different from the ket that satisfies those conditions at a later time. People working with the Schrödinger picture assume that there is a ket that satisfies the vacuum conditions at all times, but such a ket does not exist. However, the vacuum ket at a particular time is a well-defined concept in the Heisenberg picture.

I am now going to make two assumptions for the physical interpretation of the theory.

Assumption 1. Each physical state corresponds to a q-number K which is a constant of the motion in the Heisenberg picture. K will involve the time explicitly, as well as the dynamical variables at time t.

$$\frac{dK}{dt} = 0 \qquad \text{or} \qquad i\frac{\partial K}{\partial t} + KH - HK = 0.$$

In fact, we have been working with such a K all along, during our integration of the Heisenberg equations of motion.

I would like you to note that this assumption differs very drastically from the usual assumption. It assumes that a physical state corresponds to a q-number and not to a ket vector.

Assumption 2. Anything that can be observed for the state K at time t is described by $K|O_t\rangle$.

With this second assumption the normal ordering of K becomes important. This normal ordering involves putting all emission operators in any term to the left of all absorption operators. After the normal ordering has been carried out, those terms of K that contain any absorption operators will contribute nothing when K is multiplied into $|O_t\rangle$. According to assumption 2, these terms will be latent at time t. The only things that show up are those terms in K involving only emission operators.

Each of those terms will refer to certain particles in certain definite states. We can take the coefficient of each such term and form the square of its modulus. We then get a definite number and we can call that number an "intensity" for the appearance of these particles in the corresponding states at time t. I use the word *intensity* here and not the usual word *probability*, and there is a reason for that. These intensities cannot be normalized; or if you do normalize them at one time they will not remain normalized. Of course, they cannot remain normalized, because any latent terms that there are in K at one particular time may cease to be latent at a later time and contribute to observable things.

This interpretation does not give such definite results as the usual interpretation with the Schrödinger theory because we do not have things we can definitely nail down as probabilities. We can only say that if the intensity is large, the probability of observing the particles is large; if the intensity is zero, the probability is zero.

You will note that there is less determinism in the present theory than there is in the usual theory involving the Schrödinger wave function. The reason for this is that the quantity $K|O_t\rangle$, which gives all that is observable at one particular time, does not fix what is observable at a later time, because of the latent terms in K—latent terms in K which you could provide with any coefficients you like and they will not contribute anything observable at time t, but they will still influence observable things at another time.

In this theory there is no S-matrix. The S-matrix is something that is unitary and gives probabilities. I just do not know how to define an S-matrix working with the Heisenberg picture. Maybe future developments will show some way of introducing an S-matrix, but the usual way of introducing it in field theory involves so much departure from logic that I do not see how one could take it over into a logical theory.

The quantum field theory, which I shall leave in this state, is therefore far from being complete. There are various further ways in which one needs to develop it. One of the most important of these, of course, is to find some way of dealing with strong interactions. Electrodynamics works fairly well, because there the coupling constant is small and we can set up a perturbation theory in terms of it, but for strong interactions the coupling constant is not small, and one cannot use the Hamiltonian theory with this kind of perturbation method. The main problem for the future development of the Hamiltonian theory is to find some method of approximation that will enable one to apply the Heisenberg picture to collision problems when the coupling constant is not small. There are many physicists who believe that the Hamiltonian theory breaks down for these problems; I do not think that there is any evidence for this breakdown; instead I think that the breakdown is only in the perturbation method.

There is another unsolved problem in connection with electrodynamics. I have done the one-electron calculations starting off with a zero-order approximation consisting of the emission operator of one electron. One might do a corresponding calculation for a one-photon problem, starting off with the zero-order approximation consisting of the emission operator of one photon.

Well, if one does that one runs into trouble. One gets a self-energy for the photon which is quadratically infinite. If we bring in the cutoff it means we have the self-energy involving g^2, an extremely large self-energy for the photon. This is incorrect, because, among other things, it violates gauge invariance. People usually just say that this self-energy should not be there and ignore it for that reason.

People have worried a good deal about how, starting with a gauge-invariant theory, one can get a result that is not gauge-invariant. But from what I said before that is not at all surprising, because in these one-particle problems we use a perturbation method in which we assume that a quantity which is not gauge-invariant is small. There we have an assumption that is not gauge-invariant. It is surprising that this assumption works so well for one-electron problems, and it is not surprising that it fails for one-photon problems.

§32. *Relationship of the Present Field Theory to the Usual One*

The usual formulation of quantum field theory makes extensive use of the concepts of ingoing fields and outgoing fields. The connection between these two sets of fields provides what is called the S-matrix. Let us examine how these fields are defined, to get an opinion of the value of this method.

We define free fields to be fields that satisfy the field equations in the absence of interaction. Thus a free field quantity, B say, satisfies

$$\frac{i\hbar dB}{dt} = BH_0 - H_0B, \tag{32.1}$$

where H_0 is the sum of the energies of all the fields separately, without any interaction.

We may consider the free fields which are equal to the actual fields of the Heisenberg picture at some standard time T. This fixes them completely. Let us denote such free fields at time t by B_{tT}. Thus B_{TT} equals the actual Heisenberg field at time T.

The free field variables B_{tT} for a given t and T must satisfy the same commutation (or anticommutation) relations as the variables B_{TT}, since the B_{tT} are connected with the B_{TT} by a unitary transformation built up from the infinitesimal unitary transformations described by (32.1). Thus the B_{tT} satisfy the same commutation relations as the Heisenberg variables at a given time. This result holds for any interaction energy, even one with a cutoff.

Let us suppose the variables B_{tT} go to definite limits as $T \longrightarrow -\infty$. These limiting values are the ingoing fields. They can be expressed as the actual fields minus retarded fields, the latter being similar to those that occur in classical electrodynamics.

Of course it cannot be strictly true that the B_{tT} tend to definite limits as $T \longrightarrow -\infty$, because the B_{tT} will always be subject to substantial variations as T varies, but for T very large and negative it may well be that the variations are of so remote a character that they do not affect the calculations one is interested in. Under these conditions the ingoing fields become well-defined concepts. The outgoing fields are the corresponding quantities with $T \longrightarrow \infty$.

If the ingoing fields are well defined they are very useful. Their Fourier amplitudes are constants of the motion and may be taken as the basic variables in terms of which one integrates the Heisenberg equations. They have simple transformation laws under Lorentz transformations, so they lead to the possibility of the theory being set up in a manifestly covariant form.

However, in the case of electrodynamics the ingoing fields are in general not well defined. One can see this by taking the simple example of two electrons scattering one another, the force between them being mainly the Coulomb force. When the electrons are far apart the wave functions representing them do not tend to plane waves, because the Coulomb force falls off too slowly with increasing distance. The ingoing waves for this scattering problem have definite frequencies and directions, but not definite phases, so they are not completely definable.

For fields involving forces of interaction that fall off more rapidly than the inverse square law this difficulty does not arise. One can proceed to build up the theory in terms of the Fourier amplitudes of the ingoing fields. These Fourier amplitudes will consist of operators η of creation of ingoing particles and the conjugate operators $\bar{\eta}$ of annihilation of ingoing particles. One can keep to the usual physical interpretation of having the states correspond to kets, a state at all times corresponding to a ket in the Heisenberg picture. The vacuum state at all times then corresponds to the ket $|0\rangle$ satisfying

$$\bar{\eta}|0\rangle = 0$$

for all $\bar{\eta}$, and a general state at all times corresponds to the ket $f(\eta)|0\rangle$, the function f being some power series.

One cannot set up a general quantum electrodynamics on these lines. One must avoid the use of ingoing fields. The present book shows how this can be done. One is forced to a new physical interpretation, but it is really not so different from the above one, because the K defining a state at all times takes the place of the $f(\eta)$ above, both of them being constants of the motion in the Heisenberg picture.

It is, of course, a severe handicap not to be able to use ingoing fields. In particular, one is prevented from setting up the theory in a manifestly covariant form. Maybe future progress will consist in making some generalization of the concept of ingoing fields so that it is applicable even to quantum electrodynamics.

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